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THE ELEMENTS OF PHYSICS

VOL. I

MECHANICS AND HEAT

·The M Co·

THE

ELEMENTS OF PHYSICS

A COLLEGE TEXT-BOOK

BY
eamington
EDWARD L. NICHOLS
AND
WILLIAM S. FRANKLIN

IN THREE VOLUMES

VOL. I

MECHANICS AND HEAT

THIRD EDITION, REWRITTEN, WITH ADDITIONS

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PREFACE.

The first edition of this elementary treatise on physics was issued in 1896 and 1897.

The first volume, *Mechanics and Heat*, was rewritten in 1898; the second volume, *Electricity and Magnetism*, was rewritten in 1901; and the third volume, *Light and Sound*, was rewritten in 1902.

In this second revision of the volume on Mechanics and Heat the following chapters have been rewritten, namely, Chapter II. on Physical Quantity, Chapter III. on the Laws of Motion, Chapter VI. on Elasticity and Friction, Chapter XI. on the Properties of Gases, and Chapter XII. on Thermodynamics, and minor changes have been made in nearly every part of the book. A chapter on the Transfer of Heat has been added, an appendix on the Applications of Thermodynamics is planned, and a carefully selected list of problems is given.

With this second revision of the first volume, the book is finished.

The authors' thanks are due to Mr. C. M. Crawford for many valuable suggestions and for assistance in reading proof.

The authors beg the indulgence of their patrons in the omission, from this imprint of their *Mechanics and Heat*, of the Appendix on the *Applications of Thermodynamics* as scheduled in the table of contents and as called for by several references in the text.

South Bethlehem, Pa.,
September 15, 1903.

266732

EXTRACT FROM PREFACE TO FIRST EDITION.

THE beginner in Physics, if he is to be well taught, must study a text-book which contains concise statements of physical laws and a systematic development of principles. He must attend lectures or demonstrations in which the phenomena are shown. and finally he must enter the laboratory and make physical measurements.

The present book deals only with the first part of this triple course of instruction.

The authors have thought it best to face squarely the difficulties of the subject, and they may perhaps be permitted to say with Tait: * "He who expects to find this work, elementary as it is, everywhere easy reading, will be deservedly disappointed. No branch of science is free from real and great difficulties even in its elements. Any one who thinks otherwise has either not read at all or has confined his reading to *pseudo*-science."

What follows the word *Law*, as used in this text, is in every case an independent statement of experimental fact. Many physical laws are the direct result of experiment, others, such as Newton's Laws of Motion, the Laws of Thermodynamics, etc., are in their accepted form more or less indirectly derived from experiment, and are so comprehensive as to defy complete experimental verification. The principle that the only proper discussion of a law is that which is neceasary to make its meaning clear has been closely followed. The theoretical considerations which follow the statement of a number of laws referring to the same thing, for example the discussion of the Kinetic theory, which follows the statements of the various laws of gases, have in every case the object of forming concrete pictures of the abstractions contemplated in the laws. Helmholtz * has long since pointed out the advantages of this method of treatment.

---

* Tait, Heat p. viii of the preface.

The work is issued in three small volumes, the first of which treats of Mechanics and Heat, the second of Electricity and Magnetism, and the last of Sound and Light.

ITHACA, NEW YORK, Dec. 3, 1895.

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* "Es hat grosse Vortheile für das sichere Verständniss solcher Abstractionen, wenn man sich möglichst concrete Bilder davon zu machen sucht, selbst wenn diese manche Voraussetzung hineinbringen die für das Wesen der Sache nicht gerade nothwendig ist." — von Helmholtz, Handbuch der physiologischen Optik.

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CHAPTER I.

MEASUREMENT OF LENGTH, TIME, AND MASS.

1. Length: the meter and the yard. — The *Meter* is the distance, at the temperature of melting ice, between the ends of a platinum bar preserved in the Archives of France. The centimeter, which in physics is adopted as a fundamental unit, is one one-hundredth of a meter. This bar was made by Borda to be the ten-millionth part of the distance from the equator to the pole of the earth. The measurements upon which it was based are now known to be distinctly incorrect, which fact, together with the fact that copies* of the standard meter can be made with greater accuracy and convenience by direct comparison with Borda's bar than by comparison with the earth's quadrant, has resulted in the adoption of Borda's bar as the ultimate standard meter.

The *Yard* is defined by act of Parliament as the distance at 62° F. between the transverse lines in two gold plugs in a bronze bar deposited in the office of the Exchequer in London. The yard is equal to 0.914383 meter.

2. Measurement of Length: Scale and Vernier. — Lengths are commonly measured by means of divided scales. The measurement is accomplished by counting the number of scale divisions

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*See Nature, Vol. 51, p. 420, Feb. 28, 1895, for a description of the *Prototypes Internationales* of the meter and of the kilogram.

which are included in the length to be measured. Fractions of a division may be estimated by the eye or, if great accuracy is desired, fractions of a division may be determined by means of a device called a *vernier*. The general principle of the vernier is as follows: The divided scale is represented by S, Fig. 1. Let us call the divisions on this scale millimeters for brevity. A short auxiliary scale V, the vernier, is $(n - 1)$ millimeters long and it is divided into n equal parts. The diagram, Fig. 1, is constructed for $n = 10$. Let the space f, equal to a/n mm., be the fraction of a millimeter to be determined; f' is $1/n$ mm. shorter, f'' is $2/n$ mm. shorter and so on so that the a^{th} mark on the vernier is coincident with a mark on the scale. Thus a is determined. The number of the mark on the vernier which is coincident with a mark on the scale is the numerator and the number of divisions on the vernier is the denominator of the fraction which expresses the space f in terms of a scale division. The position of the zero mark of the vernier on the scale is called the *reading of the vernier*. Thus the reading of the vernier in Fig. 1 is 18.7 mm.

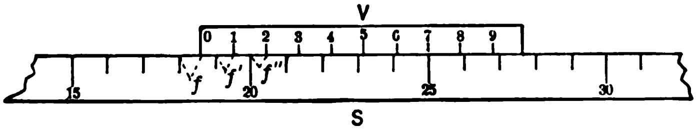

Fig. 1.

To measure a given length, any chosen point on the vernier is made to coincide with one end of the length and the vernier reading is taken. The vernier is then moved until the same point of the vernier coincides with the other end of the length and the vernier reading is again taken. The difference of these two vernier readings is the distance the vernier has been moved and it is equal to the given length. This is exemplified in the use of the cathetometer as described in the next article.

Remark. — The vernier is frequently used to determine fractions of divisions on divided circles.

3. The cathetometer (Fig. 2) is an instrument for measuring differences in level. It consists of a vertical scale upon which is a sliding piece, with clamp, carrying a telescope accurately leveled by means of a sensitive spirit level, and a vernier which plays over the vertical scale. The telescope is sighted at one of the two points of which the difference in level is required and the reading of the vernier taken. The telescope is then sighted at the other point and the vernier reading again taken. The difference of these readings is the required difference in level.

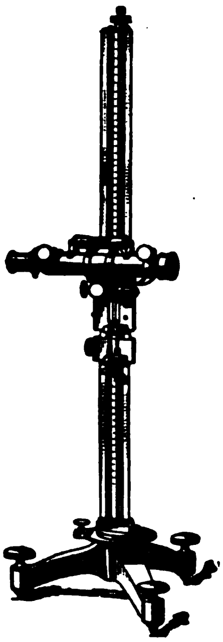

Fig. 2.

4. The dividing engine: the micrometer screw. — For more precise measurement of length, instruments are used which depend for their action upon properties of the screw.

The *dividing engine* is such an instrument. It consists essentially of an accurate horizontal screw* having a divided circular head for estimating fractions of a turn. This screw engages a nut which moves a carriage. This carriage carries a reading microscope or a graving tool, and slides on a heavy metal bed which extends to one side of the sliding carriage as a platform. *The standard of length is placed upon this platform parallel to the screw, and the screw is turned until the microscope sights exactly at one end of it. The screw is then turned, a turns, until the microscope sights at the other end of the standard. The bar of which the length is to be determined is put in place of the standard, the screw is turned until the microscope sights at one end of the bar and then, b turns, until the microscope sights at the other end of the bar. Then the length of the bar is to the length of the standard as b : a.*

Lengths up to a meter may be measured by the dividing engine with an accuracy of about one ten thousandth part of a centimeter.

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* See Encyclopedia Britannica, 9th ed., article *Screw.*

In the manufacture of divided scales an unruled bar is placed upon the platform of the dividing engine, the screw is turned until the graving tool is conveniently near to one end of the bar, and a mark is made; the screw is then turned, $\frac{a}{n}$ turns, and another mark is made, and so on, thus dividing the bar into *n*ths of the standard length. In use the dividing engine must be maintained at a constant temperature.

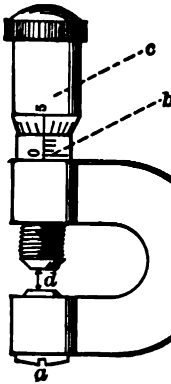

Fig. 3.

A screw which is used in the measurement of very short lengths is called a *micrometer screw*. In such instruments as the *micrometer caliper* and the *spherometer*, which are extensively used, the screw is so cut that one turn may represent a convenient (integral) fractional part of the unit length, so that turns of the screw may be read off directly as units of length. In this way the double operation described in the discussion of the dividing engine, is avoided.

The *micrometer caliper* is a micrometer screw mounted as shown in Fig. 3. The instrument is so adjusted that the distance *d* is indicated directly by the screw, the whole turns upon a fixed scale *b*, and the fractions of a turn upon the circular head *c*.

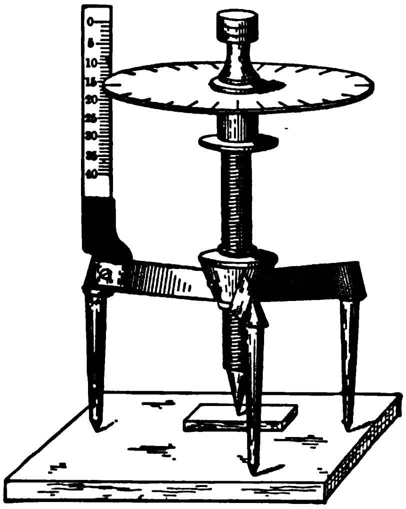

Fig. 4.

The *spherometer* (Fig. 4) is a micrometer screw, the nut of which is carried on a rigid metal tripod so that the screw may be perpendicular to a plane upon which the tripod stands. The screw carries a conical tip, the apex of which is equidistant from the feet of the tripod, and these are equidistant from each other. The instrument is used for determining the thickness of plates or the radii of curvatures of spherical surfaces. For the latter purpose a reading is taken when the screw is just in contact with a plane surface upon which the instrument stands, and again when it is placed upon the given spherical surface. From these readings, the pitch of the screw and the distance between the feet of the tripod being known, the radius of the surface is easily calculated.

5. Angle. — Angles are ordinarily specified in degrees. It is often more convenient, however, to express angle in terms of $\frac{\text{arc}}{\text{radius}}$. The unit angle in this system is called the *Radian*, and it is the angle of which the arc is equal to the radius.

Measurement of Angles. — In many instruments angles are measured by means of a divided circle. This is placed with its center at the apex of the angle, which is measured by counting the divisions on the circle between the lines which determine the angle. These lines are established by using a pair of sights, or a telescope, fixed to an arm, called an alidade. The latter turns on a pivot at the center of the circle, and carries a vernier playing over the divided circle. Frequently, two verniers situated 180° apart are used to eliminate errors due to inaccurate centering of the pivot.

Indirect measurement of length and angle: triangulation. — Consider a triangle ABC (Fig. 5), of which the side b, only, is accessible. The lengths of the sides a and c and the angle B may be calculated by trigonometry when the side b and the angles A and C have been measured. This method for determining inaccessible lengths is called *triangulation.* It is much used in surveying and in astronomy.

Poggendorff's method for measuring angles. — In many instruments it is necessary to measure the angle through which a suspended body is turned. For this purpose a small mirror, mm

(Fig. 6), is fastened to the suspended body so as to turn with it. A straight scale, *ss*, is placed at a measured distance, *d*, in front of and parallel to the mirror. A telescope, *t*, which establishes a sight line, is placed so that the scale is seen through it in the mirror, the sight line being perpendicular to the scale. The reading, *a*, of this sight line on the scale is taken; the mirror then turns through the angle θ into the position $m'm'$, and the scale reading, *b*, is again taken. The sight line is deflected through the angle 2θ, so that $\frac{a-b}{d} = \tan 2\theta$, from which the angle θ may be calculated.

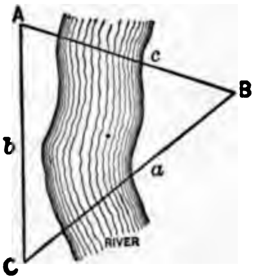

Fig. 5.

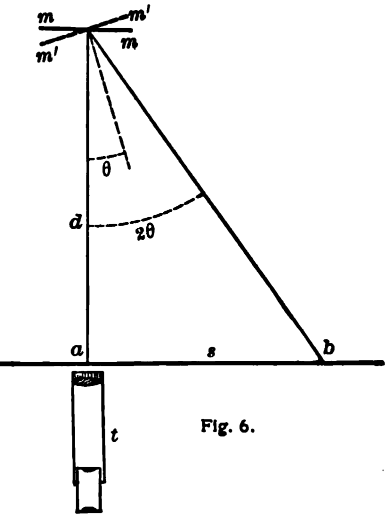

Fig. 6.

6. Area. — The unit of area is defined as the area of a square* of which the side is of unit length.

Measurement of area. — Area is determined fundamentally by calculation from measured linear dimensions.

The *planimeter* is an instrument for measuring irregular plane areas; strictly for determining the ratio of two such areas, for the instrument is standardized by measuring with it a regular area of which the area is known by calculation from measured linear dimensions.

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* The *circular mil*, much used by electricians as a unit area, is the area of a circle one *mil* ($\frac{1}{1000}$ inch) in diameter. The area of any circle in circular mils is equal to the square of its diameter in mils.

Theory of the planimeter. — Consider a line AB (Fig. 7) of length l, moving in any manner in the plane of the paper. The motion of the line may at each instant be considered as compounded * of a motion of translation and a motion of rotation, with angular velocity $\frac{d\theta}{dt}$, about an arbitrary point p distant D from the center of the line. The area swept by this moving line is considered positive when the line sweeps over it from left to right to an observer looking from A towards B. Let v be the resolved part, perpendicular to the line, of its velocity of translation. The line sweeps over area at the rate lv, because of its motion of translation, and at the rate $lD\frac{d\theta}{dt}$, because of its motion of rotation, so that the total rate at which the line sweeps area is at each instant:

$$\frac{dA}{dt} = lv + lD\frac{d\theta}{dt}. \tag{i}$$

Let a wheel, radius r, mounted at p, with its axis parallel to AB, be allowed to roll on the paper as the line moves, and let $\frac{d\psi}{dt}$ be the angular velocity of rolling of the wheel. Then $v = r\frac{d\psi}{dt}$, and equation (i) becomes

$$\frac{dA}{dt} = lr\frac{d\psi}{dt} + lD\frac{d\theta}{dt}, \tag{ii}$$

or †

$$A = lr\psi + lD\theta, \tag{iii}$$

in which A is the total area swept by the line during the time that the wheel has turned through the angle ψ and the line has turned about p through the angle θ.

---

* See Article 90

† See Article 28.

If the line comes back to its initial position, or parallel thereto, so that $\theta = 0$, then equation (iii) becomes

$$A = lr\psi. \qquad \text{(iv)}$$

That is the total area swept by AB is proportional to the angle ψ turned by the rolling wheel, and the circumfrence of the wheel may be so divided as to read areas directly.

Let one end of AB (Fig. 8), be constrained to move along a branch AC of any curve, while the other end passes once around a closed line D. Any area outside of D which is swept over by the line AB at all is swept as many times to the right as to the left, and all parts of D are swept once more to the right than to the left, so that the total area swept by AB is equal to the area of D. In its simplest form the planimeter consists of an arm AB with a rolling wheel. The end A is constrained to move on the arc of a circle, being hinged to one end of an auxiliary arm, the other end of which is fixed by a pivot.

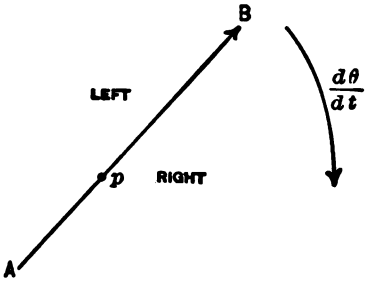

Fig. 7.

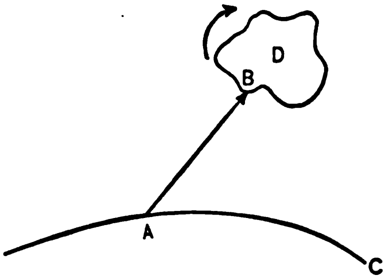

Fig. 8.

7. Volume. — The unit of volume is defined as the volume of a cube of which the edge is of unit length.

Measurement of volume. — Volume is determined fundamentally by calculation from measured linear dimensions. Thus the volume of a rectangular parallelepipedon is equal to the product of its length, breadth, and height.

Volumes of liquid and of grain are usually measured by means of a vessel of known volume. The *graduate* is a vessel upon the side of which is a scale from which the volume of a portion of liquid may be read off; it is much used by apothecaries and by chemists. A method for determining volume by weighing is described under density.

8. Mass. — Materials may be measured *by volume*, *by weight*, or *by mass*. The volume and weight of a given body are variable, but the mass of a given body is *invariable*; therefore the mass of a body is the best measure of the amount of material in the body.

The precise significance of the terms *weight* and *mass* is explained in article 39.

Units of mass. — The *Kilogram* is the mass of a piece of platinum deposited in the Archives in Paris. The gram, which has been adopted by common agreement as the fundamental unit of mass, is one one-thousandth of a kilogram. This kilogram was made by Borda to be equal to the mass of 1000 c.c. of pure water at 4° C., but copies * of the kilogram can be made with much greater accuracy and with greater convenience by direct comparison with the "Kilogramme des Archives" than by comparison with the cubic decimeter of water, so that the "Kilogramme des Archives" is taken as the standard.

The Pound (avoirdupois), which is the British commercial unit of mass, is the mass of a piece of platinum deposited in London. The pound (avoirdupois) is equal to 0.453593 kilogram.

9. Measurement of mass. *The balance.* — The analytical balance is the instrument commonly used for the precise comparison of masses. The essential parts which are shown in Fig. 9, consist of a very delicately mounted equal-arm lever with suspended pans, in which the bodies to be compared are placed. The position of rest of the pointer p with loaded pans is the same as with empty pans if the masses compared are equal. *The balance is used simply for indicating the equality of the masses of two bodies.*

Set of weights. — The determination of the mass of a body by means of the balance depends upon the use of a set of weights

* See Nature, Vol. 51, p. 420.

which may be combined in such a way as to match the mass of the body. Such a set of weights may be made by taking two bodies weighing together one kilogram and equalizing them, then two bodies weighing together one-half a kilogram and equalizing them, and so on. A set of weights more convenient in use is one which contains a five, a two, and two ones, of each — units, tens, hundreds, etc., of grams.

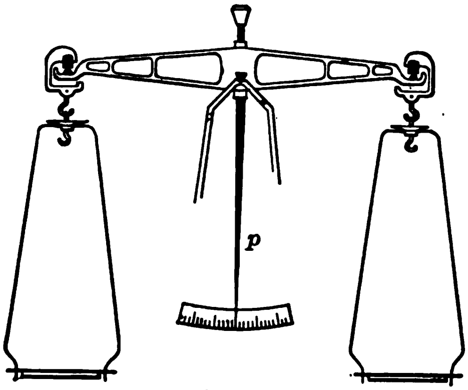

Fig. 9.

Weighing by swings. — The mass of a body which is being weighed cannot, as a rule, be precisely matched by combining the various weights of a set. Having matched the mass of the body as nearly as possible, the slight outstanding difference between the body and the weights may be determined as follows: Let E (Fig. 10) be the position of rest of the pointer with empty pans, and L the position of rest of the pointer when the body is on one pan and weights W are on the other. Let L' be the position of rest of the pointer when a known small weight w is added to W. Then the weight which must be added to W to match the body precisely is $w \cdot \frac{L-E}{L-L'}$. In practice the points E, L, and L' are determined by starting the balance swinging and observing an odd number of elongations of the pointer. The

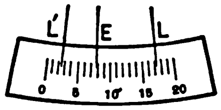

Fig. 10.

position of rest lies midway between the mean of the elongations to the right and the mean of the elongations to the left. This is called *weighing by swings.*

Errors in the use of a balance. — The indications of a balance are subject to error because of (*a*) inequality of arms, (*b*) difference in buoyant force of the air upon the body and upon the weights, and (*c*) errors of weights.

Errors due to inequality of arms are eliminated by weighing the body first on one pan and then on the other. As the result of this operation we have $b = \sqrt{W_r W_l}$ in which b is the true mass of the body, and W_r and W_l are the apparent masses when weighed upon the right and left pans respectively.

Proof. — Since r and l are the length of the right and left arms of the balance respectively, we have from the principle of the lever $W_r l = br$ and $W_l r = bl$; from which by elimination of $\frac{r}{l}$ we have $b = \sqrt{W_r W_l}$.

The errors due to the buoyant force of air are easily eliminated when the density, δ, of the weights; the density, Δ, of the body; and the density, λ, of the air, are known. Let b be the apparent mass of the body and W its mass independently of errors due to buoyant force of air. Then

$$W = b - \frac{b\lambda}{\delta} + \frac{b\lambda}{\Delta}$$

In this expression, $\frac{b}{\delta}$ is the volume of the weights, and $\frac{b\lambda}{\delta}$ is the buoyant force of the air upon the weights, by which amount the mass of the body is overestimated. Similarly, $\frac{b\lambda}{\Delta}$ is the buoyant force of the air upon the body by which amount the mass of the body is underestimated.*

10. Errors of Weights. — Consider a set of weights A, B, C, D, and E of nominal values 500 gr., 200 gr., 100 gr., 100 gr., and

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* For further discussion of the theory of the balance, and for descriptions of various forms of balances and scales, see Violle, Cours de Physique, I., p. 245; also Encyclopedia Brittanica, 9th ed., article *Balance.*

100 gr., respectively. Having a small weight of which the mass is known approximately, we weigh A in terms of B, C, D, and E, whence

$$A = B + C + D + E + a$$

similarly by repeated weighing we obtain

$$B = C + D + b$$
$$C = D + c$$
$$D = E + d$$

Having a standard kilogram, we weigh it also; whence we obtain

$$1000 = A + B + C + D + E + e$$

From these five equations, A, B, C, D, and E, which are the only unknown quantities, may be calculated.

11. Density. — The *density* of a body is defined as its *mass per unit volume.* This definition may be expressed by the equation

$$D = \frac{M}{V} \qquad (1)$$

in which D is the density, M is the mass, and V is the volume of the body.

The *specific gravity* of a substance at a given temperature is

$$S = \frac{D}{d} \qquad (2)$$

in which S is the specific gravity of the substance, D is its density, and d is the density of water.

Since masses of equal volumes of a substance and of water are proportional to their densities, we have also

$$S = \frac{M}{W} \qquad (3)$$

in which M is the mass of a portion of the substance, and W is the mass of the same volume of water.

12. Measurement of density. — The density of a substance is determined fundamentally by weighing a measured volume of the substance. Thus Borda (and, since his time, others) determined the density of water as follows: A glass cube of which the volume was calculated from its measured linear dimensions was weighed and then submerged in water and weighed again. The loss of weight * of the cube is equal to the weight (mass) of its volume of water. The mass of a known volume of water is thus determined and the density of the water may be calculated from equation (1). The density of water at a given temperature being thus determined, its density at other temperatures may be determined by methods described in Chapter IX.

Measurement of specific gravity. — The specific gravity of a substance is determined by weighing equal volumes of the substance and of water. The specific gravity may then be calculated by equation (3). When the specific gravity of a substance has been determined, its density may be calculated by equation (2), the density of the water being known.

The specific gravity of a solid is easily determined by weighing it and then submerging it in water and weighing again. The loss of weight is the weight of its volume of water.

In case of a liquid a bottle may be used to measure off equal volumes of the liquid and of water for weighing. The most convenient method for determining the specific gravity of a liquid is by means of the hydrometer.

13. The hydrometer. — The ordinary hydrometer (Fig. 11) is a glass float with a vertical cylindrical stem. This float sinks to different depths in liquids of different specific gravities, and upon the stem is a scale which indicates the specific gravity of the liquid in which the instrument floats.

For the construction of a *specific gravity* scale, the water-mark w (Fig. 12) and the mark m, to which the instrument sinks in a liquid of known specific gravity a, are determined, and the distance l between these marks is measured. The distance d from the water-mark to the mark m', to which the instrument will sink in a liquid whose specific gravity is s, is

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* The word *weight* cannot always be used in its proper sense without awkward circumlocution. The verb *to weigh* indeed signifies either the finding of the *mass* of a body or the finding of the *weight* (proper) of a body.

$$d = l\,\frac{1-\frac{1}{s}}{1-\frac{1}{a}} \tag{4}$$

from which the position of each mark of the scale may be calculated. The figure shows such a scale for liquids between sp. gr. 1.00 and sp. gr. 1.50.

Proof of equation (4). — A floating body displaces its weight of liquid. The volume of water displaced by the instrument being considered unity, the displaced volume of the liquid of sp. gr. a is $\frac{1}{a}$ and the displaced volume of a liquid of sp. gr. s is $\frac{1}{s}$; therefore the volume of the stem between water-mark and a-mark is $1-\frac{1}{a}$, and the volume of stem between water-mark and s-mark is $1-\frac{1}{s}$. Since the stem is cylindrical, these volumes are proportional to the distances l and d, whence the formula.

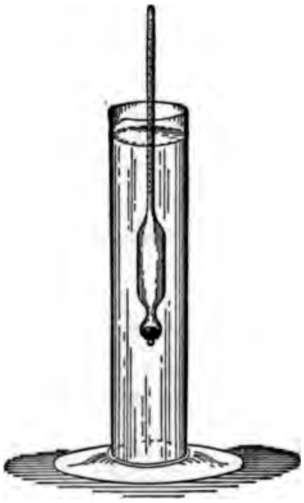

Fig. 11.

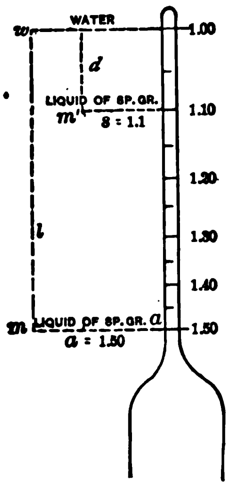

Fig. 12.

The hydrometer may be graduated so as to indicate percentage strengths of solutions of a given substance as follows: The specific gravities of solutions of the substance of various known percentage strengths are determined; these specific gravities are substituted in succession for s in equation (4), and the marks so located are numbered as percentages.

Beaumé hydrometer scales. — Various arbitrary scales of equal parts have been proposed for hydrometers. Of these scales those of Beaumé are most extensively used. Beaumé's scale for heavy liquids is constructed by locating the water-mark (near the top of the stem), and the mark to which the instrument sinks in a 15% solution of common salt. The space between these marks is divided into fifteen equal parts, and divisions of like size are continued down the stem. These marks are numbered downwards from the water-mark. A liquid is said to have a specific gravity of 26° *Beaumé heavy* when the hydrometer sinks in it to mark number twenty-six on this scale.

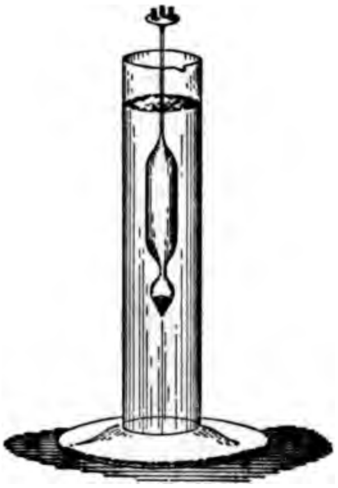

Fig. 13.

Beaumé's scale for light liquids is constructed by locating the mark to which the instrument sinks in a 10% solution of common salt (near the bottom of the stem), and the water-mark. The space between these marks is divided into ten equal parts, and divisions of like size are continued up the stem. These divisions are numbered upwards from the salt solution mark. A liquid is said to have a specific gravity of 17° *Beaumé light* when the hydrometer sinks in it to mark number seventeen on this scale.

Fahrenheit's hydrometer is a float with a narrow stem which supports a pan upon which weights may be placed. It is shown in Fig. 13. Let I be the weight of the instrument out of water, w the weights needed on the pan to sink the instrument in water to a certain mark, and l the weights needed to sink the instrument to the same mark in a liquid. Then the specific gravity of the liquid is $\frac{I+l}{I+w}$.

14. *Gravimetric method for measuring volume.* — The volume of a vessel may be determined by weighing the water or mercury which it will hold, and then calculating the volume of the water or mercury by equation (1). The vessels used by chemists for measuring volume are standardized in this way. The volume of a solid may be determined by weighing it, then submerging it in water and weighing again. The difference is the mass of its volume of water, and its volume may then be calculated from equation (1).

15. Time. — The *second*, which is the accepted unit of time in physical measurements, is defined as the 86400th part of a mean solar day.

Measurement of time. — Any movement of a body which repeats itself in equal intervals of time is called *periodic motion;* single movements are called *vibrations.* All methods, with unimportant exceptions, for measuring time depend upon periodic motion. A vibrating *pendulum* is the most familiar example. *The number, a, of vibrations in a day, and the number, b, in the interval to be measured, are counted. The interval is then* $\frac{b}{a}$ *of a day.* A *clock* is simply a machine for maintaining and counting the vibrations of a pendulum. In portable clocks a *balance wheel* takes the place of a pendulum.

Anything which affects the time of vibration of a pendulum leads to erroneous values for time intervals as measured by a clock. The time of vibration of a pendulum is affected (*a*) by temperature, on account of increase of length of pendulum with temperature; (*b*) by variations of atmospheric pressure, on account, mainly, of variation of buoyant force of air with pressure; (*c*) by variation in amplitude of vibration; (*d*) by irregularities in the manner in which impulses are imparted to the pendulum by the clockwork. The influence of temperature is avoided by using what are called *compensated* pendulums, which do not change their effective length with temperature. The variations due to (*c*) are in part obviated by providing constant driving power, which requires the gears and escapement to be of fine workmanship. The variations due to (*d*) are obviated by using what are called *dead beat* escapements, which impart their impulse to the pendulum at the instant when it passes through the vertical position.*

16. The Chronograph. — The determination of a time interval by means of a clock requires the clock reading to be taken at the beginning and at the end of the interval. Practice enables an observer to take the clock reading at the instant of a given signal accurately to a tenth of a second, with an approximately constant "personal" error which does not greatly affect the value of the interval. In this method, which is called the *eye and ear* method, the observer looks for the signal and listens to the beats of the clock.

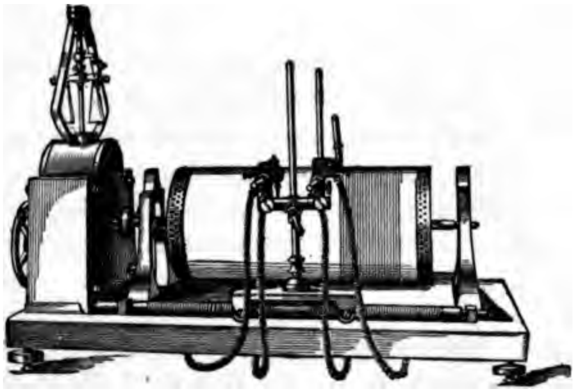

Fig. 14.

The chronograph (Fig. 14) is an instrument for enabling clock readings to be taken with greater ease and accuracy than is possible by "eye and ear." A pen traces a line upon a uniformly moving strip of paper. This pen is fixed to the armature of an electro-magnet, which is excited at each beat of the clock by an electric current controlled by a contact device actuated by the clock pendulum. A kink is thus made in the traced line at each beat of the pendulum. At the instant for which the clock reading is desired, the electro-magnet is momentarily excited by pressing a key which closes an auxiliary electric circuit, thus making an extra kink in the traced line, and the clock reading is determined by measuring off the position of this extra kink among the kinks produced by the beats of the pendulum.

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* For discussion of errors, and description of escapements, see Encyclopedia Britannica, 9th ed., article *Clock*. For description of compensated chronometer balance and chronometer escapement, see Lockyer's book entitled *Star-gazing*, pp. 175 to 210.

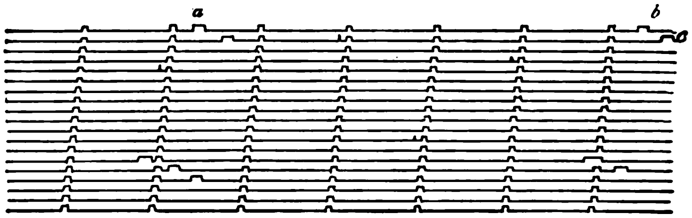

Fig. 15.

The above description applies to the essential features of the chronograph. The form of the instrument as ordinarily used for accurate time observations is shown in Fig. 14. A large sheet of paper is wrapped around a cylinder which is rotated at a uniform rate by clock work. The tracing pen and electro-magnet are mounted on a sliding carriage which is slowly moved parallel to the axis of the rotating cylinder by a screw which is driven by the same clock-work that drives the cylinder. The pen thus traces a helical line on the paper-covered cylinder, so that the large sheet of paper is equivalent to a very long narrow strip.

Fig. 15 shows a reduced facsimile of a portion of the paper sheet upon which a chronographic record has been made. The regularly spaced kinks are those produced by the beats of the clock, and the kinks marked *a*, *b*, *c*, etc., are those produced by depressing the key.

CHAPTER II.

PHYSICAL QUANTITY.

17. Measures ; units. — In the expression of a physical quantity two factors always occur, a numerical factor and a unit. The numerical factor is called the *measure* of the quantity. Thus a certain length is 65 centimeters, a certain time interval is 250 seconds, the mass of a certain body is 200 grams, a certain electric current is 26 amperes.

Arithmetical or algebraic operations among physical quantities involve both *units* and *measures*. Thus the area of a rectangle is equal to the product of its length into its breadth, say 10 centimeters × 15 centimeters which is equal to 150 centimeters × centimeters or 150 *square centimeters*. The velocity of a body is the quotient obtained by dividing the distance traveled in a given interval of time by the duration of the interval, say, 150 meters ÷ 25 seconds which is equal to 6 meters/seconds or 6 *meters per second*. The word per indicates that the unit following is in the denominator. In general we have

$$am \times bn = ab \cdot mn$$

and

$$am/bn = a/b \cdot m/n$$

in which a and b are numerical factors, and m and n are physical units. In every such case the expressions mn and m/n are *new physical units*.

Remark. — It is important to carry the units through with every numerical calculation, the arithmetical operations among the various units being indicated algebraically. When this is done there can be no ambiguity as to the meaning of the result, and when this is not done the result has strictly no physical meaning at all.

18. Units, fundamental and derived. The *fundamental* physical units are those which are fixed by arbitrary preserved standards. Thus the unit of length is preserved as a platinum bar in Paris, the unit of mass is preserved by a piece of platinum in Paris, and the second is inherently preserved in the constancy of speed of rotation of the earth.

Derived physical units are those which are defined in terms of the fundamental units and of which no material standard need be preserved. Thus the unit of area is defined as the area of a square of which each side is a unit of length, and there is no need of preserving a material standard of the unit of area. The unit of velocity is defined as unit distance traveled per second and there is no need of preserving a material standard of the unit of velocity, indeed, it would be impracticable to preserve a velocity.

Remark 1. — Quantities such as area, volume, velocity, electric current, etc., for which derived units are used, may be called *derived quantities* for the reason that they are defined (as quantities) in terms of the *fundamental quantities* length, mass, and time. For example, the density of a body is defined as the ratio of its mass to its volume; the velocity of a body is defined as the quotient obtained by dividing the distance traveled during an interval of time, by the interval, etc.

Remark 2. — There is much latitude in the choice of fundamental units. A single fundamental unit would be theoretically sufficient, inasmuch as it is possible to define all physical units in terms of any one. The choice of fundamental units is a matter which is governed solely by practical considerations; in the first place the fundamental units themselves must be easily preserved as material standards, and in the second place the fundamental quantities must be susceptible of very accurate measurement, for the definition of a derived unit cannot be *realized* with greater accuracy than is represented in the measurement of the fundamental quantities as involved in the definition.

The system of units at present in vogue presents one or two cases in which a derived quantity can be measured with far greater accuracy in terms of an arbitrary unit than the value of the arbitrary unit can be determined in terms of the units of length, mass and time. Thus electrical resistance can be measured in terms of an arbitrary unit, such as the resistance of a given piece of wire, with far greater accuracy than the value of the arbitrary unit can be determined in terms of the theoretical derived unit of resistance.

In all cases where the measurement of a derived quantity in terms of length, mass, and time is laborious or inaccurate, such measurement is carried out once for all with extreme care in the making of a material copy of the derived unit, which copy is preserved and used as a standard of reference. Thus the electromotive force of the Clark standard cell has been very carefully determined in terms of the theoretical derived unit of electromotive force, and this cell is used as a reference standard in accurate measurements of electromotive force.

19. The c. g. s. system of units. — Derived units based upon the *centimeter* as the unit length, the *gram* as the unit mass, and the *second* as the unit time, are in common use. This system of derived units is called the c. g. s. system.

Thus the square centimeter is the c. g. s. unit of area, the cubic centimeter is the c. g. s. unit of volume, one gram per cubic centimeter is the c. g. s. unit of density, one centimeter per second is the c. g. s. unit of velocity, etc.

Secondary derived units. — In many cases the c. g. s. unit of a quantity is either very large or very small in comparison with the ordinarily occurring values of the quantity. In such cases it is convenient to use a multiple or a submultiple of the c. g. s. unit. Such units are called *secondary* units or *practical* units. Thus the work required to heat a gram of water one degree is 42,000,000 c. g. s. units (ergs) or 4.2 joules. The power developed by a given engine is 1,500,000,000 ergs per second or 150 watts, the electrical resistance of an ordinary incandescent electric lamp is

220,000,000,000[*] c. g. s. units of resistance or 220 ohms. The electrostatic capacity of an ordinary Leyden jar is 0.000,000,000,000,000,005[*] c. g. s. units of electrostatic capacity or 0.005 micro-farads.

20. Dimensions of derived units. — The definition of a derived unit always implies an equation which involves the derived unit together with one or more of the fundamental units of length, mass, and time. This equation solved for the derived unit is said to express the dimensions of that unit. Thus the velocity of a body is defined as the quotient l/t where l is the distance traveled by the body during the interval of time t, so that the unit of velocity is equal to the unit of length divided by the unit of time.

Examples. — Let l be the unit of length, m the unit of mass, and t the unit of time. Then the unit of area is equal to l^2, the unit of volume is equal to l^3, the unit of density is equal to m/l^3, the unit of velocity is equal to l/t, the unit of force is equal to ml/t^2, the unit magnetic pole is equal to $\sqrt{ml^3}/t$, etc.

Remark 1. — In those cases in which a derived unit of the c. g. s. system has no specific name it is specified by its algebraic expression in terms of the *centimeter*, the *gram*, and the *second*. Thus the unit of momentum is written gr. cm./sec. (spoken *gram-centimeter per second*). In case of units which have complicated dimensions this method is not convenient in speech. Thus we specify a certain magnetic pole as 150 gr.$^{\frac{1}{2}}$ cm.$^{\frac{3}{2}}$/sec., spoken 150 *c. g. s. units pole*. Many derived units have received specific names. Such are the *dyne*, the *erg*, the *ohm*, the *volt*, etc.

Remark 2. — When the units are carried through a numerical calculation as mentioned in article 17, the unit in terms of which the result is expressed is known. Thus the distance traveled by a body may be calculated as the product of the velocity of the body by the time of travel, for example, a velocity of 200 cm./sec. multiplied by 20 seconds gives 4000 centimeters.

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[*] In the writing of very large or very small numbers it is more convenient and more intelligible to use a positive or negative power of 10 as a factor. Thus 220,000,000,000 is best written as 22×10^{10}, and 0.000,000,000,000,000,005 is best written as 5×10^{-18}.

The work done on a body by a force may be calculated by multiplying the force by the distance the body has moved in the direction of the force, and the unit of work in terms of which the result is expressed is indicated by the product of the unit of force into the unit of distance, whatever these units may be. Thus 200 dynes of force multiplied by 100 cm. equals 20,000 dyne-cm. of work or 20,000 ergs. The momentum of a moving body may be calculated by multiplying the mass of the body by its velocity. Thus 200 grams multiplied by 10 centimeters per second equals 2000 gram-cm. per second of momentum.

Although the unit in terms of which a result is expressed is known when the units are carried through a numerical calculation, it frequently happens that the unit is so entirely novel that it might almost as well be unknown. Thus the rule for finding the area of a rectangle by taking the product of length and breadth is entirely general no matter what units of length are used, and the area of a rectangle 2 meters long and 50 centimeters wide is equal to 2 meters × 50 centimeters or 100 meter-centimeters. Now the *meter-centimeter* is a unit of area equal to the area of a rectangle one meter long and one centimeter wide and it is so entirely unfamiliar as a unit of area among men engaged in practical work that one might almost as well not know the value of an area at all as to have it given in terms of such a unit. *It is for this reason nearly always necessary to reduce the data of a problem to certain units before these data can be used intelligibly in numerical calculations.*

21. Scalar and vector quantities. — A *scalar quantity* is a quantity which has magnitude only. Thus everyone recognizes at once that to specify 10 cubic meters of sand, 25 kilograms of sugar, 5 hours of time, is in each case a complete specification. Volume, mass, time, energy, electric charge, etc., are scalar quantities.

A *vector quantity* is a quantity which has both magnitude and direction and to specify a vector one must give both its magnitude and direction. This necessity of specifying both the magnitude and direction of a vector is especially evident when one is concerned with the relationship of two or more vectors. Thus if one travels a stretch of 10 kilometers and then a stretch of 5 kilometers more, he is by no means necessarily 15 kilometers from home; his position is in fact indeterminate until the direction of each stretch is specified. If one man pulls on a car with a force A of 200 units and another pulls with a force B of 100 units, the total force acting on the car is by no means necessarily equal to 300 units. In fact the total force is unknown both in magnitude and direction until the direction as well as the magnitude of each force A and B is specified. Length, velocity, acceleration, momentum, force, magnetic field intensity, etc., are vector quantities.

Representation of a vector by a line. — In all discussions of physical phenomena which involve the relationships of vectors it is a great help to the understanding to represent the vectors by lines. Thus in the discussion of the combined action of several forces on a body, it is a great advantage to represent each force by a line. To represent a vector by a line draw the line in the direction of the vector (from any convenient point) and make the length of the line proportional to the magnitude of the vector. Thus if a northward velocity of 600 centimeters per second of a moving body is to be represented by a line, draw the line to the north and let each unit length of line represent a chosen number of units of velocity.

When a vector α is represented by a line, the line is parallel to α and the value of α is given by the equation

$$\alpha = S \cdot l \qquad (5)$$

in which l is the length of the line and S is the number of units of α represented by each unit length of the line. The quantity S is called the *scale* to which the line represents the vector α.

22. Addition of vectors. — Many cases arise in physics where it is desired to consider the *single force* which is equivalent to the combined action of *several forces* on a body; where it is desired to consider the *single actual velocity* which is equivalent to *several* *velocities* of a body each produced, it may be, by a separate cause; where it is desired to consider the *single actual intensity* of a magnetic field due to the combined action of several causes each of which alone produces a magnetic field of a definite intensity and direction, etc. The single actual vector is in each case called the *vector sum* of the several vectors. Only vectors which are alike in kind can be added. Consider a number of like vectors which are represented by the lines $\alpha, \beta, \gamma \cdots$. Let the line α be drawn from any point O, Fig. 16, let the line β be drawn from the extremity of α, the line γ from the extremity of β, etc. Then *the line drawn from O to the extremity of the last line represents the sum of the several vectors.*

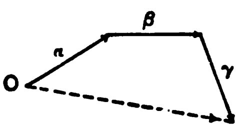

Fig. 16.

Remark. — The sum of a number of like vectors is zero when the several vectors are parallel and proportional to the respective sides of a closed polygon, and in the direction in which the respective sides of the polygon would be traversed in going round the polygon.

23. Resolution of vectors. — Any vector may be replaced by a number of vectors of which it is the sum. The simplest case is that in which a vector is replaced by two vectors which are parallel and proportional to the respective sides of a parallelogram, of which the diagonal represents the given vector. If a rectangle be constructed whose diagonal represents a given vector, then the sides of the rectangle will represent what are called the *rectangular components* of that vector in the direction of those sides.

Example. — Consider a force F (Fig. 17), pulling on a car which is constrained to move along a track. The force F is equivalent to α and β combined. The force α has no effect in moving the car, so that β is the effective part of F. This force β is called the *resolved part* of F, or the *component* of F in the direction of the track.

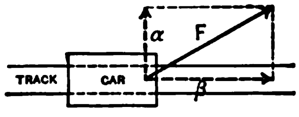

Fig. 17.

24. Scalar and vector products and quotients. — The product or quotient of a scalar and a vector α is another vector parallel to α.

Examples. — The distance l traveled by a body in time t is equal to the product vt, where v is the velocity of the body; and l is parallel to v.

The force F with which a fluid pushes on an exposed area a is equal to the product pa where p is the hydrostatic pressure of the fluid (a scalar). The vector direction of an area is the direction of its normal and this is parallel to F.

A force F which does an amount of work W in moving a body a distance d (which is parallel to F) is equal to W/d.

The acceleration of a body is equal to $\Delta v/\Delta t$ where Δv is the increment of velocity in the time interval Δt; the acceleration is a vector which is parallel to Δv.

25. Vector Products.

CASE I. *Parallel vectors.* — The product or quotient of parallel vectors is a scalar. Thus $W = Fd$, in which W is the work done by a force F acting through a distance d in its direction: $p = \frac{F}{a}$, in which p is the pressure in a liquid which exerts a force F on an exposed area a; $V = la$, in which V is the volume of a prism of base a and altitude l.

CASE II. *Orthogonal vectors.* — The product of two mutually perpendicular vectors is a third vector at right angles to both factors. Thus $a = lb$ and $b = \frac{a}{l}$, in which a is the area of a rectangle of length l and breadth b; $T = Fl$, in which T is the *moment* or *torque* of a force F, and l is its arm. The product of a vector and a line perpendicular thereto is called a *moment* of the vector.

CASE III. *Oblique vectors.* — The product of two oblique vectors consists of two parts, one of which is a scalar and the other is a vector. Consider two vectors, α and β (Fig. 18). Resolve β into two components, β' and β'', respectively parallel to and perpendicular to α. Then

$$\alpha\beta = \alpha(\beta' + \beta'') = \alpha\beta' + \alpha\beta'',$$

in which $\alpha\beta'$ is a scalar and $\alpha\beta''$ is a vector. The scalar part of a vector product is indicated thus, $S.\,\alpha\beta$ (read *scalar-alpha-beta*). The vector part of a vector product is indicated thus, $V.\,\alpha\beta$ (read *vector-alpha-beta*). When $V.\,\alpha\beta = 0$, α and β are parallel; when $S.\,\alpha\beta = 0$, α and β are orthogonal.

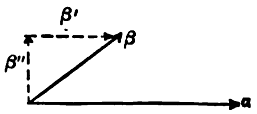

Fig. 18.

Examples.—(1) *V. bl* is the area of any parallelogram of which *b* and *l* are the sides. (2) *S. al* is the volume of any parallelopiped of which *a* is the area of the base and *l* is the other edge.

26. Constant and variable quantities. — In the study of those physical phenomena which are steady we are chiefly concerned with physical quantities which are either constant or which change so slowly that the phenomena do not depend to any perceptible extent upon the changes which are taking place at each instant. In the study of such phenomena the attention need not be directed to what is taking place at particular instants of time inasmuch as the phenomena persist unchanged throughout long intervals of time.

In the study of rapidly varying phenomena, such as the phenomena presented by a weight twirled on a string, we are chiefly concerned with physical quantities which are subject to rapid changes. In studying such phenomena *the attention must be directed to what is taking place at this or that instant of time*, or, as it is more correctly expressed, *to what takes place in very short intervals of time*, or, to borrow a term from the modern photographer, one must make mental snap-shots as it were of the varying conditions.

The most important notion in connection with a changing quantity is what is called its *rate of change*.

The *average rate of change* of a quantity x during a time interval Δt is defined as the quotient obtained by dividing the actual change Δx suffered by that quantity during the interval, by the interval. The *instantaneous rate of change* of x is the limiting value of this quotient when the time interval is very short. The instantaneous rate of change of x is sometimes represented by the symbol $\dot{x}$, sometimes by the symbol dx/dt. In case $\dot{x}$ is variable, its rate of change is called the *second rate of change* of x. It is represented by the symbol $\ddot{x}$, or by the symbol d^2x/dt^2.

27. Graphical representation of a changing vector. — Consider a changing vector α (it may be, for example, the varying velocity of a tossed ball). Imagine a line l, Fig. 19, to be drawn from a fixed point O so as to represent α *at each instant* to scale S. Then

$$\alpha = S \cdot l \tag{i}$$

The line l is of course thought of as changing its length and direction so as to be always parallel and proportional to α, and as α changes the end P of the line will move. Let V be the velocity of P, then

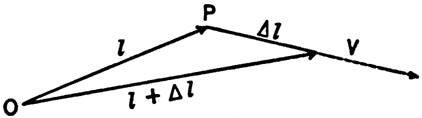

Fig. 19.

$$\dot{\alpha} = S \cdot V \tag{ii}$$

That is, *the velocity of the moving end of the line represents* $\dot{\alpha}$ *to the same scale to which the line represents* α.

Proof. — After time interval Δt elapses the vector will have become $\alpha + \Delta\alpha$, which value of the vector will be represented by the new value of the line $l + \Delta l$, so that

$$\alpha + \Delta\alpha = S \cdot (l + \Delta l) \tag{iii}$$

Subtracting equation (i) from (iii), member by member, we have

$$\Delta\alpha = S \cdot \Delta l \tag{iv}$$

and dividing both members of this equation by Δt, we have

$$\frac{\Delta\alpha}{\Delta t} = S \cdot \frac{\Delta l}{\Delta t} \tag{v}$$

Now $\Delta\alpha/\Delta t$ is $\dot{\alpha}$ and $\Delta l/\Delta t$ is the velocity V. Therefore

$$\dot{\alpha} = S \cdot V \qquad \text{Q. E. D.}$$

28. General Propositions concerning Rates of Change.

(*a*) Consider a quantity x which changes at a *constant rate* $\dot{x}$. The change which x suffers in any time interval t is $\dot{x}t$.

(*b*) Consider a quantity y which always changes a times as fast as x. If the two quantities start from zero together, then y is always equal to ax. If y already has the value C when x is zero, then y is always equal to $ax + C$.

Conversely, if one quantity is always a times as large as another, it must always change a times as fast.

(*c*) Consider a quantity s which is equal to the sum of a number of variable quantities x, y, z. Let Δx, Δy, and Δz be the increments of x, y, and z, respectively, during an interval Δt, then the increment of s is

$$\Delta s = \Delta x + \Delta y + \Delta z \tag{i}$$

Dividing both members of this expression by Δt, we have

$$\frac{ds}{dt} = \frac{dx}{dt} + \frac{dy}{dt} + \frac{dz}{dt} \tag{ii}$$

That is, *the rate of change of the sum of a number of quantities is equal to the sum of the rates of change of the respective quantities.*

NOTE. — If the relation (ii) is given, that is, if s is a quantity whose rate of change is known to be equal to the sum of the rates $\dot{x}$, $\dot{y}$, and $\dot{z}$, then s is equal to $x + y + z$ provided that s, x, y, and z start from zero together.

29. Distributed quantity.[*]—A physical quantity may refer to the state of a medium, having definite values at each point, in such a way that the value of the quantity at a point specifies the state of the medium at that point. Such a quantity is called a *distributed quantity*. Thus in a liquid the pressure has at each point a definite value (distributed scalar), and a moving liquid has at each point a definite velocity (distributed vector). A magnetic field, in the same way, has at each point a definite intensity (distributed vector). Another example is that of the *strain* at each point in a distorted solid, which is definite in value.

The distribution of a quantity is said to be *homogeneous* or *uniform* when it has the same value at all points; otherwise the distribution is said to be *non-homogeneous*. In the study of phenomena dependent upon distributed quantity, in cases in which the distribution is non-homogeneous, *the attention must be directed to what takes place in small regions.*

30. Complex physical quantity. — A scalar may be defined as a quantity which depends upon a single numerical specification. A vector depends upon three independent numerical (scalar) specifications. There are quantities which are more complex. Thus *stress* and *strain*, in the simplest case, each depends upon six independent numerical (scalar) specifications, and there are other physical quantities of still higher complexity. All such complex quantities are intimately connected with geometry, and it is very important that the student attain to a degree of familiarity with them, inasmuch as the imaging faculty is greatly strengthened thereby. For purposes of numerical calculation scalars only are important. For such purposes all complex quantity must be degraded to scalars, and the scheme made use of in this degradation must be kept in mind.

Methods for reaching numerical results by vector calculations can only be attained by geometrical mechanical appliances as exemplified in graphical statics. Arithmetical methods are essentially scalar and require all data to be reduced to scalars.

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[*] See chapter XVII, Vol. II., for a full discussion of distributed quantity.

CHAPTER III.

FORCE; LAWS OF MOTION.

31. Force and its effects. — Our fundamental notions of force arise from the sensations which we associate with muscular effort. The effect of a force upon a body may be to break the body, to distort it, to heat it, or to change its state of motion; or the effect of a force may be to change the physical state of a substance, as, for example, the liquefaction of a gas, or to produce chemical action. The fact is that in nearly every physical phenomenon force actions of one kind or another are involved.

We are here concerned chiefly with the effect a force has in changing the state of motion of a body. This effect of force was first precisely described by Sir Isaac Newton.

A force can be measured only in terms of its effects. The effect which can be most easily employed in the measurement of force is the effect the force has in distorting a body, as, for example, in stretching a helical spring (compare article 40). *The simplest effect of a force, however, is the change which it produces in the state of motion of a body, inasmuch as this effect is the same for all substances;* this effect of a force is now universally adopted as the effect by which a force is measured.

32. Translatory motion. Material particle. — There are many distinct types of motion such as *translatory** motion, *rotatory** motion, the *relative motion of the parts of a body* which is being distorted, the obscure motion which constitutes the *electric current,* etc. In studying the behavior of bodies under the action of force * we shall consider each of these types of motion by itself.

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* A body performs translatory motion when all lines in the body remain unchanged in direction. A body performs rotatory motion when a certain line in the body remains stationary, or, in general, when a certain point of the body remains stationary.

In the study of translatory motion it is convenient to imagine the moving bodies to be of such character that translatory motion is the only kind of motion of which they are capable. *A body so small that its rotatory motion plays no perceptible rôle in its behavior, or so small as to be sensibly concentrated at a point, is called a particle.* The earth, for example, as a part of the solar system is sensibly a particle. A body which is constrained to move without rotation may be treated as a particle in so far as its behavior under the action of force is concerned.

33. Position, specification of; Displacement. — The position of a particle is usually specified by giving its rectangular coördinates x, y, and z referred to arbitrarily chosen axes of reference. The line drawn from the origin of coördinates to a particle is called the *position vector* of the particle.

When a particle changes its position it is said to be *displaced* or to have suffered a *displacement.* The displacement of a particle is usually specified by giving the changes Δx, Δy, and Δz of the coördinates of the particle. The line drawn from the initial to the final position of the displaced particle is called the *displacement vector.*

34. Velocity.—The average velocity of a particle during an interval of time is the quotient $\Delta l/\Delta t$, in which Δl is the displacement of the particle during the interval and Δt is the duration of the interval. If the velocity of a particle is changing then the value of the velocity at a given instant is the limiting value of the quotient $\Delta l/\Delta t$ when the time interval is very short. (Compare Art. 26.)

The velocity of a particle is represented by the letter v.

The *orbit* of a particle is the path it describes during its motion.

The components of the velocity of a particle are the rates of change of the coördinates of the particle, namely, $\dot{x}$, $\dot{y}$, and $\dot{z}$ or dx/dt, dy/dt and dz/dt.

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* To each type of motion there is a corresponding type of force. Thus *linear force* corresponds to translatory motion; *twisting force* or *torque* corresponds to rotatory motion; *distorting force* or *stress* corresponds to distortion, *electromotive force* corresponds to electric current, etc.

Acceleration. — The acceleration of a particle is the rate of change of its velocity. Thus, if Δv is the amount by which the velocity of a particle changes during the time interval Δt, then $\Delta v/\Delta t$ is the average acceleration of the particle during the interval Δt. If the acceleration of a particle is changing, then its value at a given instant is the limiting value of the quotient $\Delta v/\Delta t$ when the time interval is very short (compare articles 26 and 27).

In this definition of acceleration it is important to understand clearly what is meant by the change Δv of a velocity. Let v_1, Fig. 20, represent the velocity of a particle at a given instant, and let v_2 represent the velocity of the particle after the time interval Δt has elapsed. Then the line Δv in the figure represents the change which the velocity of the particle has suffered during Δt.

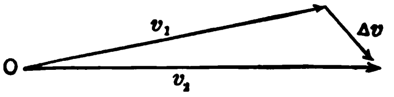

Fig. 20.

Referring to Fig. 20 it is evident that v_2 may be numerically equal to v_1, that is v_1 and v_2 may differ only in direction, and yet the increment of velocity Δv and the acceleration $\Delta v/\Delta t$ have finite values. This is exemplified in the case of the movement of a particle in a circular orbit. The velocity does not change its value in centimeters per second but changes its direction continuously and the particle has at each instant a very considerable acceleration.

The acceleration of a particle is represented by the symbol $\dot{v}$.

The components of $\dot{v}$ are the second rates of change of the coördinates of the particle, namely, $\ddot{x}$, $\ddot{y}$, $\ddot{z}$, or d^2x/dt^2, d^2y/dt^2, and d^2z/dt^2

35. Graphical representation of velocity and acceleration. — It is of the utmost importance that the student get a clear idea of velocity and acceleration. The following graphical representation will serve better than anything else to give clear ideas in this connection.

When a particle has acceleration the velocity of the particle is changing and it is necessary to represent this *changing velocity* by a *changing line.* In order that we may have a concrete case in mind let us consider the motion of a tossed ball. Imagine the line l, Fig. 21, drawn from the fixed point O', to change in direction and length so as to represent at each instant the varying velocity v of the tossed ball to scale S. Then

$$v = S \cdot l \qquad (5)$$

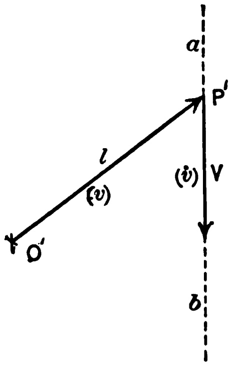

Fig. 21.

As the line l changes in direction and length the point P' will move and *the velocity V of P' represents $\dot{v}$ at each instant to the same scale* that l represents v.* That is

$$\dot{v} = S \cdot V \qquad (6)$$

Compare article 27.

As the tossed ball describes its orbit the point P', Fig. 21, will describe a certain path ab. The path traced by P' is called the ***hodograph*** to the orbit of the particle whose velocity is represented by $O'P'$; examples of hodographs are given in articles 44 and 56.

36. Newton's Laws of Motion. †

I. All bodies persevere in a state of rest or in a state of uniform motion in a straight line, except in so far as they are made to change that state by the action of force.

II. (*a*) The acceleration of a particle is parallel and proportional to the force acting upon the particle.

(*b*) The acceleration which is produced by a given force is inversely proportional to the mass of the particle.

III. Action is equal to reaction, and in a contrary direction.

(1) The first law describes the behavior of a body upon which no force acts. The behavior is simply that the *velocity of the body*

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* It should be noticed that ***units of velocity per length of line*** is of the same dimensions as ***units of acceleration per unit velocity***, otherwise the scale S could not be the same in equations (5) and (6).

† In the statement of these laws, the form given to them by Newton has been modified by the introduction of terms used elsewhere to this volume, and in all modern treatises on Physics.

does not change. When the vector sum of a number of forces which act on a body is zero, the forces are said to be balanced or in equilibrium, and the body (particle) behaves as if no force were acting upon it.

(2′) The second law describes the behavior of a body (particle) when acted upon by an unbalanced force. The behavior is simply that *the body gains velocity in the direction of the force at a rate which is (a) proportional to the force and (b) inversely proportional to the mass of the body.* When Newton expounded this fact, he had in mind a quantitative notion of force based, perhaps, upon the effect which forces have in producing distortion of elastic bodies; else he could not have affirmed that the acceleration which a force produces is *proportional* to the force. The production of acceleration is now adopted as the effect by which forces are measured, so that *one force is by definition twice as great as another if it produces double the acceleration when acting on a given particle.*

(2″) The second law may be further paraphrased as follows: (*a*) The amount of velocity gained by a given body in a given interval of time is proportional to the force acting on the body and the gained velocity is parallel to the force. (*b*) The amount of velocity produced by a given force in a given interval of time is inversely proportional to the mass of the body upon which the force acts, and the gained velocity is of course parallel to the force.

(3) The third law expresses the fact that a force is always due to the mutual action of two bodies; that this mutual action always consists of a pair of equal and opposite forces; that one of these forces acts upon body number one, and the other upon body number two.

37. Illustrations of the Laws of Motion.

A train moving on a level track at full speed (uniform velocity) is dragged backwards by the track and by the air with a force which is precisely equal to the pull of the locomotive. While starting the pull of the locomotive exceeds the drag, and while stopping the drag exceeds the pull of the locomotive.

The universal tendency of moving bodies to come to rest when left to themselves is due to the dragging forces, or friction, with which surrounding bodies act upon a body in motion. To maintain uniform motion requires these dragging forces to be balanced by an equal driving force. These dragging forces are usually greater[*] upon heavy bodies than upon light bodies. Thus a team of horses must exert a much greater pull upon a heavily loaded wagon than upon an empty wagon to keep it in motion. *The greatest obstacle, in the mind of a beginner, to a clear understanding of Newton's laws of motion is the confusion of* **the force required to balance a drag** *and* **the entirely unbalanced force required to produce acceleration.**

A body which is supported, for example, by a table, is pushed upwards by its support with a force which balances the downward pull of the earth on the body. When the support is removed, the unbalanced pull of the earth imparts velocity to the body and it falls.

When an elevator car is stationary or moving up or down at full speed, the tension of the suspending cable is equal to the weight of the car. While starting up, the pull of the cable exceeds the pull of the earth on the car (weight of car), and while starting down the pull of the earth on the car exceeds the pull of the cable. A passenger in the car stands with his normal weight upon the floor when the car is stationary or moving at full speed. While starting up the passenger stands with greater than his normal weight upon the floor, and while starting down the passenger stands with less than his normal weight.

The fact that the acceleration produced by a given force is inversely proportional to the accelerated mass is easily verified by means of Atwood's machine (Article 45). As an illustration of this fact we may imagine a given force applied to start a 20,000-ton ship and also a 2,000-ton ship. Ignoring complications arising from the drag of water and wind, the 2,000-ton ship will reach a given speed in 1/10 the time required for the 20,000-ton ship to reach the same speed.

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[*] So also are accelerating forces greater on heavy bodies than on light bodies, other things being equal. It is in this fact chiefly that the confusion, below mentioned, originates.

The water near an orifice in a tank, pushed by the water behind it, gains velocity and issues as a jet. The issuing water pushes back (reaction) upon the water in the tank, and if the tank is free it gains velocity in the direction of this reacting force.

Inertia. — The reluctance of a body to change its velocity is called *inertia.* The inertia of a body is measured by the time required for a given unbalanced force to impart to the body a given velocity. When so measured *the inertia of a body is proportional to its mass* according to Newton's second law. Inertia being thus proportional to mass may be most conveniently expressed in terms of mass. Thus we speak of a 50-ton locomotive either when we wish to specify its mass as measured on a platform scale or its inertia, as measured by its reluctance to gain or lose velocity.

38. Formulation of the second law of motion. — According to the second law of motion the force (unbalanced) which acts upon a particle is proportional to the product of the mass of the particle into its acceleration. That is,

$$F = km\dot{v} \qquad \text{(i)}$$

where F is the force acting on a particle, m is the mass of the particle, $\dot{v}$ is the acceleration of the particle, and k is the proportionality factor. This equation (i) is true whatever units are used in the expression of F, m, and $\dot{v}$.*

If we choose as a unit of force that force *which will impart unit velocity per second to unit mass*, then the value of k is unity and equation (i) becomes:

$$F = m\dot{v} \qquad (8)$$

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*For example, if force is expressed in pounds' weight (that is, in terms of the force with which the earth pulls a pound), mass in pounds, and acceleration in feet per second per second, then

$$F = \tfrac{1}{32} m\dot{v} \text{ approximately.} \qquad (7)$$

The dyne. — The force which imparts velocity to a gram at the rate of one centimeter per second per second is called the *dyne*. *The force F in equation (8) is expressed in dynes when m is expressed in grams and $\dot{v}$ in centimeters per second per second.* The dyne is a very small unit of force. The *megadyne*, or one million dynes, is sometimes used as a unit of force.

39. Weight: gravitational units of force. — The weight of a body is the force with which the earth pulls it. When a body is unsupported, the force with which the earth pulls it imparts downward velocity to it, and it falls. Let m be the mass of a body in grams, W its weight in dynes, and g the rate, in centimeters per second per second, at which it gains velocity when it falls. Then by equation (8) we have

$$W = mg. \tag{9}$$

The acceleration, g, of a falling body is called the ***intensity of gravity***. It is slightly different at different places on the earth. In middle latitudes its value is approximately $980 \frac{\text{cm.}}{\text{sec.}^2}$.

The weight of a gram and the weight of a pound are much used by engineers as units of force; these units are called the ***gram's weight*** and the ***pound's weight***, respectively. One gram's weight is approximately equal to 980 dynes. One pound's weight is equal to 453.6 × 980 dynes.

Distinction between mass and weight.—The spring dynamometer, ordinarily called a spring balance, measures the force with which the earth pulls a body. That is, such a balance determines the weight of a body, in dynes if the balance is properly calibrated, *in a definite unit of force* in any case if the spring is kept at constant temperature.

The analytical balance and the platform scale measure the force with which the earth pulls a given body *in terms of the force with which the earth pulls a standard body.* If this measurement is carried out at a place where the pull of the earth on the standard body is *large* then the pull of the earth on the given body will be *correspondingly large* so that the result of the operation is strictly independent of the pull of the earth. The analytical balance measures what is called the *mass* of a body.

40. Measurement of Force.

(*a*) *Kinetic method.*—The force acting upon a body of known mass may be calculated by equation (8), the acceleration of the body being determined by observation. This method cannot be realized in its simplicity, but it forms the basis of many physical measurements.

(*b*) *Counterpoise method.* — The force to be measured is applied at one end of a lever, and counterpoised by weights of known mass hung on the other end. The magnitude of the force is then easily calculated in gravitational measure. In case the force to be measured is the breaking force of a wire or beam, or the force necessary to produce a given elongation or flexure, it may be applied directly by means of weights of known mass.

(*c*) *By means of the spring dynamometer.* — The elongation of a spring is proportional to the stretching force. The elongation due to a known force being observed, the force which produces any observed elongation may be calculated.

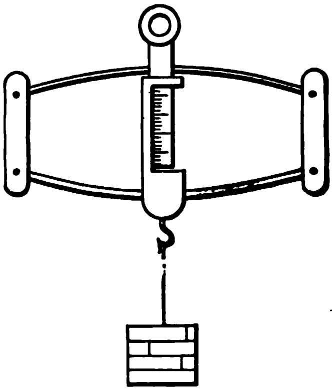

Fig. 22.

Figure 22 shows a well-known form of spring dynamometer in which flat springs are used. Such dynamometers are calibrated by applying a series of known forces and marking the deflections which they produce.

41. Uniformly accelerated motion.—*Falling bodies.* A body acted upon by an unbalanced force which is constant in magnitude and in direction gains velocity at a constant rate. Such a body is said to perform *uniformly accelerated motion.* A body falling freely under the action of the constant pull of the earth is, in so far as the friction of the air is negligible, an example of uniformly accelerated motion.

All bodies when falling freely gain velocity at one and the same rate, air friction being negligible. Thus two bricks together fall at exactly the same increasing speed as one brick alone. The doubled pull of the earth on the two bricks produces the same acceleration as the single pull of the earth on one brick. Doubling the force and doubling the mass leaves the acceleration unaltered.

Consider a particle which gains velocity at a constant rate, $g \frac{\text{cm.}}{\text{sec.}^2}$; for example, a falling body. The velocity gained in t seconds is

$$v = gt \tag{10}$$

Let v_1 be the initial velocity of the particle. Then $v_1 + gt$ is its velocity after t seconds, and its average* velocity during the t seconds is $\frac{1}{2}[v_1 + (v_1 + gt)]$ or $v_1 + \frac{1}{2}gt$; and the distance d fallen by the particle during the t seconds is equal to the product of the average velocity into the time t. That is,

$$d = v_1 t + \tfrac{1}{2}gt^2 \tag{11}$$

If v_1 is zero, equation (11) becomes

$$d = \tfrac{1}{2}gt^2 \tag{12}$$

Eliminating t between equations (10) and (12) we have

$$v = \sqrt{2gd} \tag{13}$$

which expresses the velocity of a falling body after it has fallen a distance d.

42. Projectiles. — When the initial velocity v_1 is zero, or when it is vertical, we have the ordinary case of a falling body. The addition to be performed in equation (11) is then done in the usual way. When the initial velocity v_1 is not vertical, as in case of a bullet shot from a gun, or of a ball tossed from the hand, the falling body is called a *projectile*. In this case equation (11) still holds good, but the addition, $v_1t + \frac{1}{2}gt^2$, is vector addition, and is to be performed as follows: Draw a line *OA*, Fig. 23, representing the distance v_1t to scale. This line is to be drawn in the direction of the initial velocity v_1 of the ball. From the point *A* draw the line *AB*, vertically downwards, representing the distance $\frac{1}{2}gt^2$ to the same scale. The line *OB* then represents the actual distance, and is in the actual direction of the ball from its starting-point after *t* seconds.

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* The average value, during a given interval of time, of a quantity which changes at a constant rate, is equal to half the sum of its initial and final values.

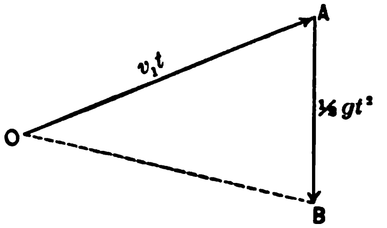

Fig. 23.

The orbit of a projectile is a parabola.

Proof. — Choose the *x*-axis of reference in the direction of v_1, and the *y*-axis, vertically downwards, as shown in Fig. 24. Then the co-ordinates of the ball after *t* seconds are

$$x = v_1t,$$

$$y = \tfrac{1}{2}gt^2.$$

Eliminating *t*, we have

$$y = \tfrac{1}{2}\frac{g}{v_1^2}x^2,$$

which is the equation to a parabola. Q.E.D.

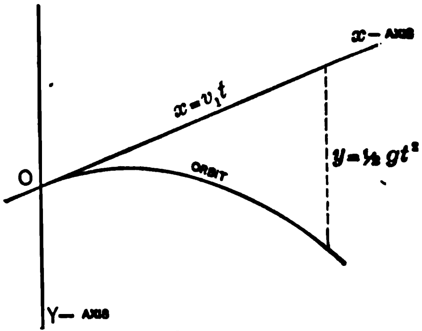

Fig. 24.

43. The range of a gun is the distance from the gun of the bullet when it reaches the level of the gun on its downward flight. The range *l* of a gun, ignoring resistance of the air, is

$$l = \frac{2\,v_1^2 \sin\theta \cos\theta}{g}, \qquad (14)$$

in which v_1 is the initial velocity of the bullet, θ is the angle of elevation of the gun, and g is the acceleration of gravity.

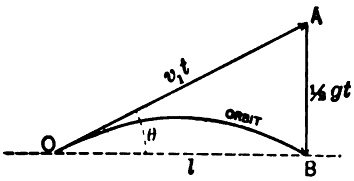

Fig. 25.

Proof. — When the bullet has reached the level of the gun, it is at *B*, Fig. 25 and *OB* is the range *l*. From this figure we have

$$v_1t\cos\theta = l,$$
$$v_1t\sin\theta = \tfrac{1}{2}gt^2.$$

Eliminating *t* from these two equations, we have equation (14). Q.E.D.

44. The hodograph to the orbit of a projectile is a vertical straight line.

Proof. — Let the line *OP*, Fig. 26, represent the initial velocity of a projectile. After *t* seconds this velocity will have been changed by the addition of a vertical velocity, *gt*, so that the line which represents the new velocity must terminate in the vertical line *AA*, which is, therefore, the required hodograph.

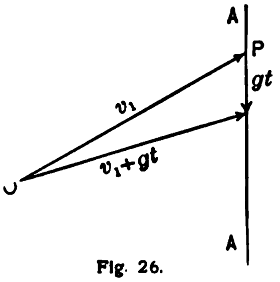

Fig. 26.

45. Experimental Study of Falling Bodies.

By the use of the inclined plane. — A body rolling down an inclined plane gains velocity at a constant rate, and therefore the distance rolled in *t* seconds is proportional to t^2. By giving to the plane a slight inclination, and especially by choosing for the body a wheel rolling on a small axle, the motion may be made slow enough to be easily followed, and the above relation verified.

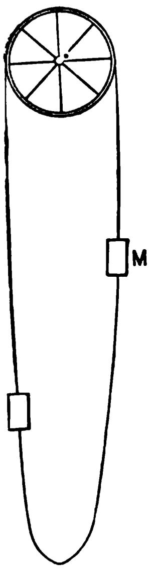

Fig. 27.

By means of Atwood's machine. —This machine consists of two weights of equal mass, *M*, suspended by a light cord over a well-balanced, light-running pulley, as shown in Fig. 27. A small weight of mass *m* is added to one side, and the unbalanced force *mg*, with which the earth pulls the added weight, causes the total mass to gain velocity at a constant rate. By choosing *m* quite small the motion may be made slow enough to be followed without difficulty.

By shooting at a falling ball. — If a gun is aimed at a ball which is released by an electro-magnetic device and allowed to fall at the instant that the projected ball leaves the muzzle of the gun, then *the projected ball always strikes the falling ball.* This is shown to be in accord with equation (11) and with the geometrical construction of Fig. 23, as follows: After t seconds the projected ball is at B, Fig. 28, at a distance $\frac{1}{2}gt^2$ below the line OA, and the falling ball is at B', likewise at a distance $\frac{1}{2}gt^2$ below the line OA. Therefore, since the balls are always at the same distance below OA, they must collide.

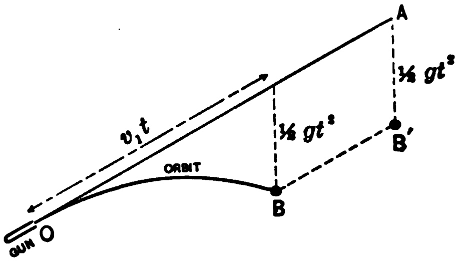

Fig. 28.

By the liquid jet. — Let JJ, Fig. 29, be the trace of a jet of water from a nozzle held in a horizontal position. This trace may be made by turning the jet carefully against a vertical blackboard. The measured lengths of the equidistant vertical lines are approximately proportional to the squares of the integers 1, 2, 3, 4, etc., which shows that the jet forms approximately a parabolic curve in accordance with Art. 42.

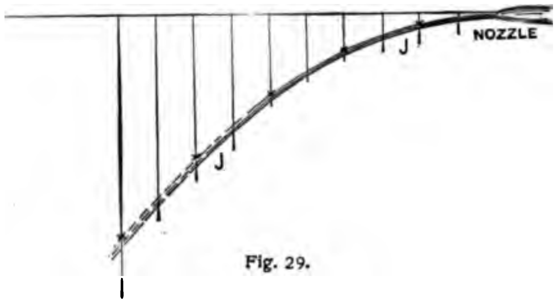

Fig. 29.

46. Projectiles in a resisting medium. — Bodies which are projected through the air *do not have constant acceleration,* because of the resistance which the air offers to their motion. In case of a heavy body moving slowly, for example, an iron ball tossed from the hand, the resisting force of the air is very small compared with the weight of the body, and the motion of the body approximates very closely to the ideal motion discussed in the foregoing paragraphs. In case of a body which is projected at high velocity the resisting force of the air is very large, and the motion of the body differs widely from the ideal motion. A heavy ball tossed from the hand describes a parabolic curve. A ball shot from a gun describes a so-called *ballistic curve*. The dotted curve, Fig. 30, represents the parabolic orbit which a projectile from a modern gun would describe in a vacuum, and the full-line curve represents approximately the curve actually described by the shot in air.

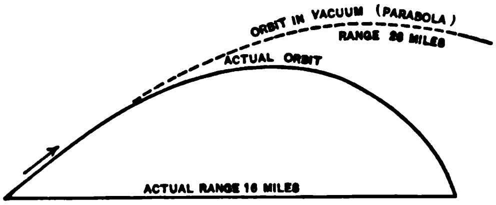

Fig. 30.

47. Universal gravitation: Newton's law. — Any two particles of matter attract each other with a force F, which is directly proportional to $\frac{mm'}{d^2}$, in which m and m' are the masses of the respective particles, and d is the distance between them. That is,

$$F = k\frac{mm'}{d^2}, \tag{15}$$

in which k is the proportionality factor. The factor k is called the *gravitation constant*.

Proposition. — The gravitational action of a spherical body on an external particle is the same as if all the matter of the sphere were concentrated at its center.

Corollary. — The mutual action of two spheres is the same as if each were concentrated at its center. Therefore, if m and m', equation (15), are the respective masses of two spheres, d the distance between their centers, then F will be their force of attraction. If this force F be observed, then k may be calculated.

48. Determination of the mass of the earth. — The force mg, with which the earth attracts a body of mass m [see equations (9) and (15)], is

$$mg = k\frac{mm'}{r^2};$$

whence

$$g = k\frac{m'}{r^2}. \tag{16}$$

In this equation m' is the mass of the earth, r is the earth's radius, and g is the acceleration of gravity. Equation (16) enables the calculation of the mass, m', of the earth when k, g, and r are known. This determination was first made by Cavendish, the difficult experimental part of his work being the observation of the extremely small force of attraction of two lead spheres of known mass at a measured distance, in order to determine the quantity, k, equation (15). The method employed by Cavendish is described in Art. 103.

Remark. — The formulation of the law of gravitation was the result of an effort on the part of Newton to find a common basis for the three laws of planetary motion previously discovered by Kepler. To test the newly formulated law, Newton applied it to the moon. His computation expressed in modern terms is as follows: The value of g at the surface of the earth is about $980\,\frac{\text{c.m.}}{\text{sec.}^2}$; and its value at the moon, according to Newton's law of gravitation, is, in round numbers, $\frac{4000^2}{240{,}000^2}\cdot 980\,\frac{\text{c.m.}}{\text{sec.}^2}$. This value of g at the moon must be identical to $\dot{v}$ of the moon in its orbit, as determined by equation (27), in which the mean radius in centimeters of the moon's orbit (240,000 miles) is written for r, and the reciprocal of the number of seconds in a sidereal lunar month is written for n. That is, n equals $\frac{1}{2{,}360{,}000}$ revolution per second.

SYSTEMS OF PARTICLES.

49. Systems of particles. — Bodies which are not small enough to be looked upon as particles are treated fundamentally* as *systems of particles.* Thus, a rotating wheel or a portion of flowing water is a system of particles. We shall speak of the collective positions of the particles of a system as the *configuration* of the system, and of the collective velocities of the particles of a system as the *velocity configuration* of the system. When we wish to speak of a phenomenon as due to either the position configuration or the velocity configuration of a system, or to both, we shall speak of it as depending upon the *state of motion* of the system.

A *closed system* is one which has no force acting upon any part of it from outside. There is no such thing in nature as a closed system, but the conception is useful nevertheless.

We shall find later that a closed system neither gives energy to nor receives energy from any other system, so that the total energy of a closed system never changes.

50. Momentum. — In the discussion of the motion of systems of particles the product, mv, of the mass of a particle into its velocity is of such importance as to warrant its receiving a name; it is called the *momentum* of the particle, it is a vector and, its direction is the same as the direction of the velocity, v, of the particle. Representing the momentum of a particle by M we have:

$$M = mv \tag{17}$$

Proposition. — Whenever an unbalanced force F acts upon a particle the momentum of the particle changes and the rate of change of the momentum, $\dot{M}$, is parallel to and equal to the force F.

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*The treatment of a rotating wheel as a system of particles leads to the conceptions of angular velocity, moment of inertia, angular momentum, torque, etc., and once these conceptions are developed the motion of a rigid body may be treated as a whole — without the need of looking upon it as a system of particles. The same is true of elastic movements, fluid motion, etc.

This is evident when we consider that M is equal to m times v and that it must therefore change m times as fast as v changes. That is $\dot{M} = m\dot{v}$; but $m\dot{v}$ is equal to the unbalanced force F, so that $\dot{M} = F$. Representing $\dot{M}$ by the symbol dM/dt we have

$$F = \frac{dM}{dt} \tag{18}$$

51. Conservation of momentum in a closed system. — The sum (vector sum of course) of the momenta of the particles of a closed system never changes.

Proof. — Consider two particles of the system. By Newton's third law these act on each other, if at all, with equal and opposite forces. By the previous article [equation (18)] the force acting on each particle is equal to the rate of change of its momentum. Therefore the mutual action of these two particles causes their respective momenta to change at equal and opposite rates so that the sum of their momenta is not affected by their mutual action. The same is true of every pair of particles in the system, and therefore the sum total of the momenta of the various particles of the system does not change.

Impact. — Consider two bodies (particles*) of which the masses are m_1 and m_2, and the velocities v_1 and v_2, respectively. The combined momentum of the two particles is $m_1v_1 + m_2v_2$. If the bodies collide, their velocities may change. Let V_1 and V_2 be the respective velocities after impact. Then $m_1V_1 + m_2V_2$ is the total momentum of the bodies after impact, and by the principle of the conservation of momentum, we have

$$m_1v_1 + m_2v_2 = m_1V_1 + m_2V_2 \tag{19}$$

Impact of inelastic balls. — When an inelastic ball, such as a ball of soft clay, strikes squarely against another, the two balls move after impact as a single body so that V_1 and V_2 are equal, and this common velocity after impact is completely determined by equation (19).

Impact of perfectly elastic balls. — Consider two elastic balls moving at velocities v_1 and v_2 in the same straight line (v_1 and

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* An actual body is called a particle when we wish to indicate that we are attending only to its translatory motion.

v_2 being opposite in sign if the balls are moving in opposite directions). Let the masses of the balls be m_1 and m_2 respectively.

When these balls collide they are distorted, at a certain instant the distortion reaches a maximum, after which the balls rebound from each other and the distortion is relieved. When the distortion of the two balls has reached its maximum the two balls are at the instant moving at common velocity c, which is determined by the equation

$$(m_1 + m_2)c = m_1v_1 + m_2v_2 \tag{i}$$

During the time that the balls are being distorted, which time we shall call the first half of the impact, the first ball loses* an amount of velocity $(v_1 - c)$ and the second ball loses* an amount of velocity $(v_2 - c)$. During the time that the balls are being relieved from distortion, which time we shall call the second half of the impact, they are assumed to act on each other with precisely the same series of forces as during the first half of the impact only in reverse order. This is what is meant by the assumption that the two balls are perfectly elastic. Therefore during the second half of the impact each ball loses the same amount of velocity as it lost during the first half of the impact, that is, the total loss of velocity by the first ball is $2(v_1 - c)$ and the total loss of velocity by the second ball is $2(v_2 - c)$, so that

$$V_1 = v_1 - 2(v_1 - c)$$

and

$$V_2 = v_2 - 2(v_2 - c)$$

or

$$V_1 = 2c - v_1 \tag{ii}$$

and

$$V_2 = 2c - v_2 \tag{iii}$$

in which V_1 and V_2 are the respective velocities of the balls after impact.

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* The velocity c lies between v_1 and v_2 so that if $(v_1 - c)$ is positive then $(v_2 - c)$ must be negative.

Substituting the value of c from equation (i) in equations (ii) and (iii) we have

$$V_1 = \frac{m_1 v_1 + 2m_2 v_2 - m_2 v_1}{m_1 + m_2} \qquad \text{(iv)}$$

$$V_2 = \frac{2m_1 v_1 + m_2 v_2 - m_1 v_2}{m_1 + m_2} \qquad \text{(v)}$$

The simplest case is where $m_1 = m_2$ and where $v_2 = 0$, that is where the balls are similar, and where the first ball only is in motion before impact. In this case the result may be derived from equations (iv) and (v) but it is more instructive to derive the result anew. The common velocity c at the middle of the impact is equal to $\frac{1}{2}v_1$. That is the first ball loses half its velocity and the second ball gains an equal amount of velocity during the first half of the impact. During the second half of the impact the first ball loses the remainder of its velocity and comes to a standstill, and the second ball gains once more an equal amount of velocity so that its velocity is now equal to the initial velocity v_1 of the first ball. That is, when an elastic ball A strikes squarely against a similar stationary ball B, the ball A stops still, and the ball B moves on with the full original velocity of A. If A is heavier than B then both balls move in the same direction after the impact. If B is heavier than A, then A moves backwards or has a negative velocity after the impact.

52. The center of mass of a system.—In so far as those phenomena are concerned which are associated with *translatory motion* of a system, or which are associated with forces which tend only to produce *translatory motion*, the entire mass of the system may be thought of as concentrated at a certain point called the *center of mass* * of the system. Thus the forces with which the earth pulls on the various parts of a body tend to produce only translatory motion and the behavior of a body under the action of gravity, *only*, is the same as if the entire mass of the body were concentrated at its center of mass.

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* Sometimes called the *center of gravity* of the system.

Example. — A body A, Fig. 31, is suspended from the ceiling by two parallel strings of the same length. Under these conditions the body A may vibrate as a pendulum *without rotation* that is performing only *translatory motion*, and the behavior of the body is the same as if it were entirely concentrated at its center of mass. This is by no means the case with an ordinary pendulum; the force which acts on an ordinary pendulum at its axis of support, together with the force of gravity, has a tendency to produce, and does produce, rotatory motion.

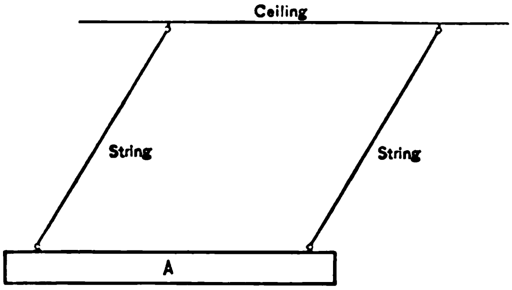

Fig. 31.

Examples of centers of mass. — The center of mass of a body of uniform density is at the geometrical center of the body. The center of mass of two particles lies on the line joining them, and the distances of the particles from the center of mass are inversely proportional to the respective masses of the particles. The center of mass of the earth and moon, for instance, is on the line joining the center of the earth and the center of the moon, and it is about 3000 miles from the center of the earth.

53. Definition of the center of mass. — Let x be the distance eastward (westward distances being considered negative) from the origin of coördinates to a particle of which the mass is Δm, let x' be the eastward distance to a particle of which the mass is $\Delta m'$ and so on.

Then the sum $x \cdot \Delta m + x' \cdot \Delta m' +$ etc. extended so as to include all the particles of a body is equal to zero if the origin of coördinates is chosen at the center of mass of the body. That is and similarly

$$\left.\begin{aligned}\Sigma x\cdot\Delta m &= 0\\ \Sigma y\cdot\Delta m &= 0\\ \Sigma z\cdot\Delta m &= 0\end{aligned}\right\}\qquad(20)$$

when the origin of coördinates is the center of mass of the body, y and z being coördinates at right angles to x and at right angles to each other.

In case the origin of coördinates is not at the center of mass of the body then the sum $x\cdot\Delta m + x'\cdot\Delta m +$ etc., divided by the total mass of the body, $\Delta m + \Delta m' +$ etc., gives the x-coördinate of the center of mass. Representing this x-coördinate of the center of mass by X we have

and similarly

$$\left.\begin{aligned}X &= \frac{\Sigma x\cdot\Delta m}{\Sigma\Delta m}\\ Y &= \frac{\Sigma y\cdot\Delta m}{\Sigma\Delta m}\\ Z &= \frac{\Sigma z\cdot\Delta m}{\Sigma\Delta m}\end{aligned}\right\}\qquad(21)$$

Remark. — The student must not imagine that equations (20) and (21) are here *derived* from the definition of center of mass given in Art. 52. It is simpler to accept equations (20) and (21) as definitions and to derive the statements of Art. 52 from them, as done in part in Art. 72 and in part in the following discussion, which, being beyond the grasp of the beginner, is given in small type with occasional important statements in large type. After the first course in mechanics is finished the student should return to this discussion of the center of mass, for nothing in the whole range of applied mechanics is of greater importance. The vector method is used in the following discussion as being by far the simplest.

54. Vector definition of the center of mass. *The mass vector.* — Consider a particle P, Fig. 32*a*, of mass m, distant l from the reference point O. The line l is the position vector of the particle. The product $ml (= R)$ is called the *mass vector* of the particle. That is:

$$R = ml$$

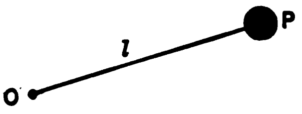

Fig. 32*a*.

The mass vector ml may be represented by a line drawn in the direction of l, and of such length as to represent the value of the product ml to a convenient scale.

Center of mass of a system. — Consider the sum of the mass vectors $m'l'$, $m''l''$, $m'''l'''$, etc., of the particles of a system. This sum is called the resultant mass vector of the system. If this resultant mass vector be divided by the total mass $m' + m'' + m''' +$ etc. of the system, the quotient is the *position vector*, L, of the *center of mass* of the system. That is,

$$L = \frac{\Sigma ml}{\Sigma m} \tag{22}$$

If the resultant mass vector, Σml, is zero, then L is zero and the reference point is the center of mass of the system. That is, when the center of mass of a system is chosen as the reference point, we have

$$\Sigma ml = 0 \tag{23}$$

This equation is equivalent to the three equations (20), and equation (22) is equivalent to the three equations (21).

Examples. — If one attempts to hold weights at one end of a horizontal bar, one recognizes the reality of the *mass vector*. Ten kilograms on a two-meter arm, five kilograms on a four-meter arm, two kilograms on a ten-meter arm, are equal mass vectors of 20 meter-kilograms.

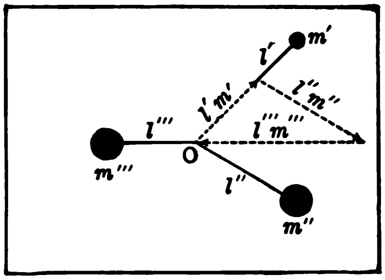

Fig. 32*b*.

If a board is hung in a horizontal position from a point O, Fig. 32*b*, in its upper surface and if weights m', m'', and m''' are placed on the board, these weights will balance each other if the sum (vector sum) of their mass vectors, reckoned from O, is zero (that is if the vectors $m'l'$, $m''l''$, and $m'''l'''$ are parallel and proportional to the sides of a triangle as shown by the dotted lines) and the point O is the *center of mass* of the weights m', m'', and m'''.

Proposition. — The momentum mv of a particle is equal to the rate of change of the mass vector of the particle. This is evident when we consider that v is the rate of change of l so that mv must be the rate of change of ml. That is

$$mv = \frac{d(ml)}{dt} \tag{24}$$

55. Motion of center of mass.

Proposition I. — The center of mass of a system upon which no outside forces act (a closed system) remains stationary or moves with uniform velocity in a straight line.

Proof. — The momentum of a particle is equal to the rate of change of its mass vector according to equation (24). Therefore we have

$$mv + m_1v_1 + \cdots = \frac{d(ml)}{dt} + \frac{d(m_1l_1)}{dt} + \cdots$$

But the sum of the rates $d(ml)/dt$, $d(m_1l_1)/dt$, etc., is equal to the rate of change of the sum $ml + m_1l_1 + m_2l_2 + \cdots$ by Art. 28. Therefore

$$mv + m_1v_1 + \cdots = d/dt\,(ml + m_1l_1 + \cdots)$$

That is, the sum of the momenta of the particles of a system is equal to the rate of change of the resultant mass vector $ml + m_1l_1 + \cdots$, of the system. But the sum of the momenta of the particles of a closed system is constant by Art. 51; therefore $ml + m_1l_1 + \cdots$ changes at a constant rate. Now the position vector of the center of mass is $L = \dfrac{ml + m_1l_1 + \cdots}{m + m_1 + \cdots}$ by equation (22), so that L also changes at a constant rate, or the end of L (the center of mass) moves with constant velocity in a straight line.

Q.E.D.

Proposition II. — The center of mass of a system of particles, upon which outside forces act, moves as would a particle of mass Σm under the action of a force equal to the sum of all the external forces which act on the various particles of the system, where Σm represents the entire mass of the system.

Proof. — As in the previous proof, we have $mv = d(ml)/dt$, whence $m\dot{v} = d^2(ml)/dt^2$. Therefore

$$m\dot{v} + m_1\dot{v}_1 + m_2\dot{v}_2 + \cdots = \frac{d^2(ml)}{dt^2} + \frac{d^2(m_1l_1)}{dt^2} + \cdots$$

$$= \frac{d^2}{dt^2}(ml + m_1l_1 + \cdots) = \frac{d^2}{dt^2}\Sigma ml$$

But $\Sigma ml = L \cdot \Sigma m$ from Art. 54, equation (22), where L is the position vector of the center of mass of the system; and since Σm is constant, we have $d^2/dt^2\ \Sigma ml = d^2L/dt^2 \cdot \Sigma m$. Therefore

$$m\dot{v} + m_1\dot{v}_1 + m_2\dot{v}_2 + \cdots = \frac{d^2L}{dt^2}\ \Sigma m$$

Now $m\dot{v}$, $m_1\dot{v}_1$, etc, are the forces acting on the particles, and the forces with which the particles act on each other occur in pairs of equal and opposite forces; therefore $m\dot{v} + m_1\dot{v}_1 + \cdots$ is the sum, F, of the *external forces* acting on the system. Furthermore d^2L/dt^2 is the acceleration, $\dot{V}$, of the center of mass, and Σm is the total mass of the system, so that

$$F = \dot{V}\Sigma m$$

Comparing this with equation (8), it is seen that the center of mass moves as described in the proposition. Q.E.D.

Examples of motion of the center of mass. *Wheel and axle.* — If the center of mass of a rotating wheel and axle lies in the axis of the shaft, it remains stationary, and therefore no outside force need act upon the shaft to keep it in place except the steady upward force necessary to support the wheel and axle against gravity.

If the center of mass is not in the axis of the shaft, then as the wheel rotates the center of mass will describe a circular path about the axis of the shaft. In this case the bearings must not only support the wheel and axle against gravity but they must exert upon the shaft the force which is requisite to constrain the center of mass to its circular path, and this force is, according to proposition II. above, equal to the force which would be necessary to constrain a particle of mass Σm to the path described by the center of mass of the wheel, where Σm represents the entire mass of wheel and axle. These constraining forces may be very great in high speed machinery, and therefore a machine part which is to rotate at high speed must be adjusted until its center of mass is in the axis of the shaft upon which the machine part rotates. This adjustment is called *balancing* and the machine part so adjusted is said to be *balanced.**

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* A machine part which is long in the direction of the shaft upon which it rotates may have its center of mass in the axis of its shaft and yet the bearings may have to

The centrifugal dryer such as is used in laundries consists of a rapidly rotating bowl with perforated sides carried on top of a vertical spindle. The materials to be dried are placed in this bowl and it is impossible to keep the bowl and contents even approximately balanced so that if the spindle were carried in a rigid bearing the machine would be disabled in a very short time because of the very great forces that would be brought into play in constraining the center of mass of bowl and contents to move in a circular path. This difficulty is obviated by supporting the spindle at the lower end, only, in a long bearing mounted on springs to hold it approximately vertical. The bowl, contents, and spindle then rotate about a line passing through their center of mass and through the center of the flexible bearing, and, although the bowl and spindle seem to wabble badly (inasmuch as they do not rotate about the axis of figure), nevertheless the machine runs quite smoothly, producing but little vibration in the supporting frame.

Planet and satellite. — If two balls which are tied together with a short string are thrown in such a way that the string is kept stretched while the balls revolve rapidly about one another, a certain point of the string will be seen to describe a smooth parabolic curve, just as a simple projectile would do. This point of the string is the center of mass of the two balls. The center of mass of the earth and the moon describes an elliptic orbit about the sun once a year, while the earth and moon rotate about their center of mass once every lunar month, in a manner very similar to the motion of the two balls just described.

exert constraining forces upon the shaft. A long cylinder loaded on opposite sides at the two ends is an example. The discussion of the motion of such non-symmetrical bodies is beyond the scope of this text. This subject is treated in a very thorough manner by Kirchhoff in his analytical mechanics.

CHAPTER IV.

CIRCULAR AND HARMONIC MOTION. STATICS. ENERGY.

56. Velocity and acceleration of a particle moving steadily in a circular orbit. — Consider a particle which makes steadily n revolutions per second in a circular orbit of radius r. The circumference of the orbit is $2\pi r$ and inasmuch as the particle traverses the circumference n times per second its velocity v is

$$v = 2\pi rn \tag{25}$$

The magnitude, or numerical value, of the velocity v is constant; but its direction is changing continuously, and, as stated in Art. 34, this continual change of direction of v involves acceleration.

To determine the acceleration of a particle which is moving steadily in a circular orbit it is necessary to consider what takes place in a very short interval of time as stated in Art. 26. At a given instant the moving particle is at P, Fig. 33. At this instant the velocity v_1 of the particle is at right angles to PO. This velocity is represented by the line $O'P'$ in Fig. 33. After the lapse of time Δt the moving particle is at Q, Fig. 33, and its velocity, now at right angles to OQ, is represented by the line $O'Q'$, Fig. 33. Then the line $P'Q'$, Fig. 33, represents the increment Δv of the velocity of the particle during the interval Δt and this increment of velocity is in the direction of PO since $P'Q'$ is parallel to PO.

The triangles OPQ and $O'P'Q'$ are similar and therefore we have

$$\frac{\Delta v}{v} = \frac{v \cdot \Delta t}{r} \tag{i}$$

in which v is written for the common numerical value of v_1 and v_2 and $v \cdot \Delta t$ is the length of the arc PQ traversed by the particle during the interval Δt.

From equation (i) we have at once

$$\frac{\Delta v}{\Delta t} = \frac{v^2}{r} \tag{ii}$$

But $\Delta v / \Delta t$ is the acceleration $\dot{v}$ of the particle, so that

$$\dot{v} = \frac{v^2}{r} \tag{26}$$

That is, a particle moving steadily in a circular orbit of radius r has a steady acceleration towards the center of the orbit (since Δv, Fig. 33, is parallel to PO) and this acceleration is equal to v^2/r, where v is the steady velocity of the particle.

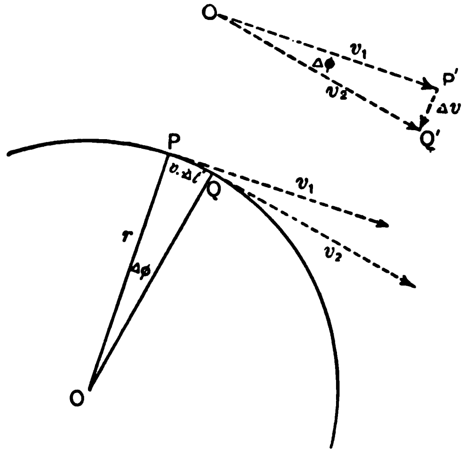

Fig. 33.

It is often convenient to have $\dot{v}$ expressed in terms of r and n. For this purpose the value of v may be substituted from equation (25) in equation (26), giving:

$$\dot{v} = 4\pi^2 n^2 r \tag{27}$$

The force required to constrain the particle to its circular orbit is, by Newton's second law, parallel to and proportional to $\dot{v}$, and by equation (8) the force is equal to $m\dot{v}$, whence using the value of $\dot{v}$ from equation (26), we have

$$F = \frac{v^2 m}{r} \tag{28a}$$

and using the value of $\dot{v}$ from equation (27), we have

$$F = 4\pi^2 n^2 r m \tag{28b}$$

where F is the force required to constrain a particle of mass m to a circular orbit of radius r, v is the steady velocity of the particle, and n is the number of revolutions per second. This force F is directed always towards the center of the circular orbit.

Remark. — The word *steady* is used in this article to indicate a velocity of which the numerical value does not change.

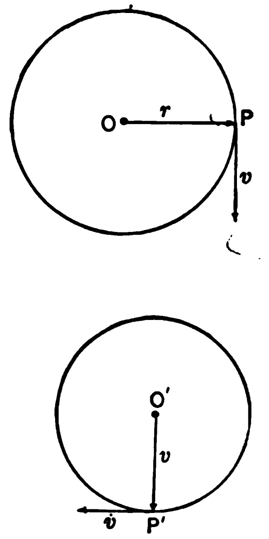

Fig. 34.

The state of affairs at each instant during the steady motion of a particle in a circular orbit is most clearly shown by the method of Art. 35, as follows:

The upper circle in Fig. 34 represents the orbit of the particle P. Imagine the changing velocity v to be represented by a changing line $O'P'$ drawn from an arbitrary fixed point O'. Now the velocity v is constant in numerical value so that the line $O'P'$ is constant in length. Furthermore, v is at each instant at right angles to the radius OP so that the point P' describes a circle, the hodograph, and the velocity of P', which is at each instant parallel to PO, represents the acceleration $\dot{v}$ of P to the same scale that $O'P'$ represents the velocity of P.

57. Examples of motion in a circle.

Consider a stone of mass m twirled on a string of length r, n revolutions per second. The string pulls on the stone with a force $4\pi^2 n^2 r m$ dynes in constraining the stone to its circular path.

It is obvious that whatever the mass of the stone, the tension of the string, which is proportional to n^2, may be increased indefinitely by increasing n.

The forces necessary to constrain the various particles of a rotating wheel to circular paths create a stress in the wheel which increases with the square of the speed (n^2), and may become great enough to burst the wheel.

A belt stretched around a pulley must press against the face of the pulley sufficiently to prevent slipping. When the belt is in motion, a portion (mv^2/r) of the force, which at rest would create pressure on the pulley face, is used to constrain the particles of the belt to their circular paths, and the belt is likely to slip. It is a familiar fact to engineers that high-speed belts are apt to slip, especially on small pulleys.

The centrifugal cream separator is a rapidly rotating bowl AB (Fig. 35), in which fresh milk is placed. The force acting on a particle of milk in this separator is $4\pi^2n^2r/g$ times as great as the weight of the particle. Therefore the forces tending to separate the cream in the separator are $4\pi^2n^2r/g$ times as great as the forces which tend to separate the cream in a gravity vat. Ordinarily n is about 100 revolutions per second, r is about 12 cm., and g is $980\frac{\text{cm.}}{\text{sec.}^2}$, so that the forces acting to separate cream in the

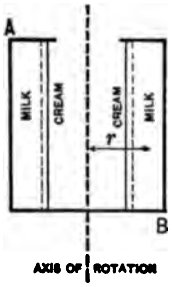

Fig. 35.

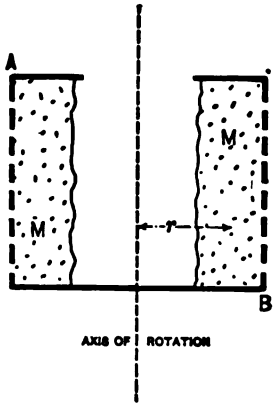

Fig. 36.

separator are about 5,000 times as great as the forces acting to separate cream from milk in a gravity vat.

The centrifugal dryer, used in laundries and in sugar refineries, is a rotating bowl AB (Fig. 36) with perforated sides, in which the material MM to be dried is placed. The forces acting to drain out the water are $4\pi^2n^2r/g$ times as great as they would be in a gravity vat with a perforated bottom. In case of ordinary laundry dryers n is about 35 revolutions per second, and r is about 20 cm., so that $4\pi^2n^2r/g$ is about 800.

A locomotive on a railway curve is an example of a body traveling in a circular path. Let us consider the elevation which should be given to the outside rail on a curve of which the radius is r, for a given velocity v of the locomotive. The thing that is desired is that the total force F (Fig. 37) with which the track pushes on the locomotive shall be perpendicular to the track (this force is of course equal and opposite to the force with which the locomotive pushes on the track). The rails are represented in section by ab, Fig. 37 and θ is the required angle.

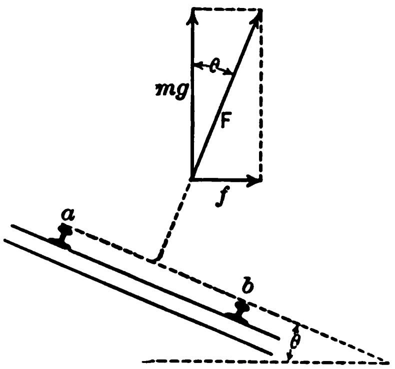

Fig. 37.

The vertical component of F is what sustains the locomotive against gravity. Therefore this vertical component is equal to mg where m is the mass of the locomotive and g is the acceleration of gravity. That is:

$$F\cos\theta = mg \tag{a}$$

The horizontal component of F is the unbalanced force which constrains the locomotive to its circular path. Therefore this horizontal component is equal to v^2m/r according to equation (28)a. That is:

$$F\sin\theta = \frac{v^2m}{r} \tag{b}$$

Dividing equation (*b*) by equation (*a*) member by member we have

$$\tan\theta = \frac{v^2}{rg} \qquad (29)$$

which determines θ in terms of the velocity v of the locomotive, the radius r of curve, and acceleration of gravity g.

58. Simple harmonic motion is the projection on a fixed straight line of a uniform motion in a circle. Consider a point P', Fig. 38, moving uniformly n revolutions per second in a circle of radius R. *The point P, which is the projection of P' on the line CD, performs simple harmonic motion.*

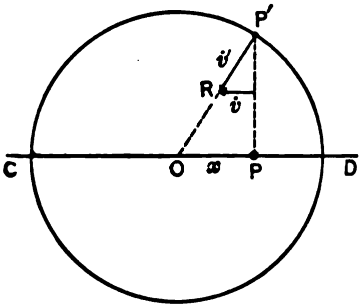

Fig. 38.

Example. — The piston of a steam engine moves in a manner which approximates to simple harmonic motion when the connecting rod is long as compared to the radius of the crank. The crank pin takes the place of the point P' in Fig. 38 and the connecting rod when long transmits the horizontal part, only, of the motion of the crank pin to the piston. A pendulum bob vibrating through a small arc performs simple harmonic motion. The prongs of a vibrating tuning fork perform simple harmonic motion. All sounding bodies which give off pure musical tones perform simple harmonic motion.

59. Definitions.

Vibration. — One to-and-fro movement of the point P, Fig. 38, is called a *vibration.*

Frequency. — The number n, of vibrations per second of the point P is equal to the number of revolutions per second of the point P', and is called the *frequency* of the vibrations.

Period. — The time τ required for one vibration of P is called the *period* of its vibrations. The relation between τ and n is

$$n = \frac{1}{\tau} \tag{30}$$

Amplitude. — The maximum value of x, namely OD, Fig. 38, is called the *amplitude* of the vibrations of P. This amplitude is equal to the radius R of the circular path of P'.

Equilibrium position. — When the vibrating particle is at O, Fig. 38, no force acts upon it, as is explained below; the point O is therefore called the *equilibrium position* of the vibrating particle.

60. Acceleration of, and force action upon, a particle in harmonic motion. — The velocity of the point P, Fig. 38, is the horizontal component of the velocity of P', and the acceleration, $\dot{v}$, of P is the horizontal component of the acceleration, $\dot{v}'$, of the point P'. From the similar triangles of Fig. 38 we have

$$\frac{\dot{v}}{\dot{v}'} = \frac{x}{R}$$

Whence, since $\dot{v}' = 4\pi^2n^2R$ from equation (27), we have

$$\dot{v} = -4\pi^2n^2x \tag{31}$$

The minus sign is taken for the reason that $\dot{v}$ is towards the left (negative) when x is towards the right (positive).

The force F which must act on the particle P to cause it to move in the prescribed manner is equal to $m\dot{v}$ by equation (8). Therefore

$$F = -4\pi^2n^2mx \tag{32}$$

in which m is the mass of the particle, n is the frequency of its vibrations, x is the distance of the particle from its equilibrium position at a given instant, and F is the force acting on the particle at that instant.

From equation (32) we have the following:

61. Proposition. — A particle which is performing harmonic motion is at each instant urged towards its middle point or position of equilibrium with a force which is proportional to its distance from the middle point at that instant. That is

$$F = -ax \tag{33}$$

where

$$a = 4\pi^2 n^2 m \tag{34}$$

Conversely, if a particle is urged back towards its equilibrium position by a force which is proportional to its displacement x from its equilibrium position, then, if started, this particle will perform simple harmonic motion.

Example. — A ball of mass m is fixed to one end of a flat strip of steel and the other end of the strip is clamped in a vise. When the ball is displaced to one side the spring is bent and the ball is urged back by a force which is proportional to the distance that the ball is displaced. If this ball is started it will perform simple harmonic motion and $4\pi^2$ times the square of the number of oscillations per second times the mass of the ball will be equal to the *constant of the spring*, that is, to the factor a which multiplied by the displacement of the end of the spring will give the force which is necessary to keep the spring bent; this force is of course equal and opposite to the force exerted by the bent spring.

Remark 1. — If we substitute $1/\tau$ for n from equation (30) in equation (34) we have

$$a = \frac{4\pi^2 m}{\tau^2} \tag{35}$$

in which τ is the period of one vibration of a particle in harmonic motion, and a and m are the same as in equations (33) and (34).

Remark 2. — The amplitude of the oscillations of a particle in harmonic motion depends upon the violence with which the vibrations are started, and whatever the amplitude may be, equation

(34) is true if equation (33) is satisfied. That is the frequency n of the vibrations is independent of amplitude if the force F is strictly proportional to x.

62. Harmonic motion and the curve of sines. — If the point P, Fig. 39, performs vertical harmonic movements of amplitude A, and if the paper be moved under P in the direction XO with uniform velocity, then the point P will trace upon the paper a *curve of sines*. This relation between harmonic motion and the curve of sines is of importance in the theory of light and sound.

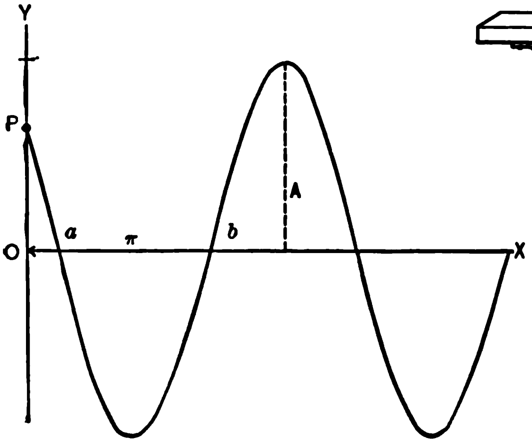

Fig. 39.

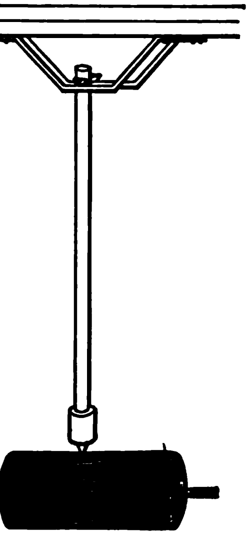

Fig. 40.

Illustration.— If a pendulum be mounted above the cylinder of a chronograph, its oscillations being parallel to the axis of the latter and in the same vertical plane as shown in Fig. 40, it may be made to trace a curve upon the smoked sheet. If the motion of the cylinder be uniform, this curve will be a curve of sines. To obtain such tracings the bob of the pendulum is

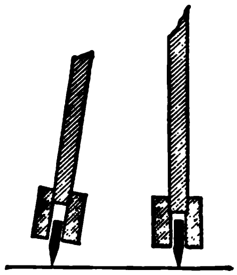

Fig. 41.

provided with a movable stylus, the point of which will remain in contact with the cylinder throughout the entire oscillation as shown in Fig. 41. Similar tracings may be obtained with vibrating rods, wires, and other sounding bodies.

STATICS.

63. The effect of a force. — The effect of a force upon a body depends upon

(*a*) The magnitude of the force;
(*b*) Its direction;
(*c*) Its point of application.

In the study of the action of forces on bodies it is convenient to distinguish two separate actions of a force corresponding to the two possible kinds of motion; namely, *translatory* motion and *rotatory* motion.

64. The tendency of a force to produce translation depends only upon the direction and magnitude of the force, and is the only effective part of a force which acts upon a material particle or upon a body constrained to move without rotation. The combined tendency of a number of forces to produce translation is that of their vector sum.

65. First condition of equilibrium. — In order that a number of forces may have no tendency to produce translatory motion of a body, it is necessary and sufficient that their vector sum be equal to zero (see Art. 22). This condition requires the sum of the components of the forces in any direction to be equal to zero; so that this first condition of equilibrium may be formulated thus:

$$\begin{aligned} \Sigma X &= 0 \\ \Sigma Y &= 0 \qquad (36) \\ \Sigma Z &= 0 \end{aligned}$$

In these equations X, Y, and Z are the components of the various forces in the directions of the axes of reference.

66. Torque. — The tendency of a force to produce rotatory motion is called *torque.** The torque action of a force is the only effective part of a force which acts upon a rigid body of which one point is fixed. Let F (Fig. 42) be a force acting at the point p upon a body, of which the point O is fixed. The torque action of this force is

$$T = Fl \qquad (37)$$

in which l is the perpendicular distance from O to the line of action of the force, as shown in the figure.

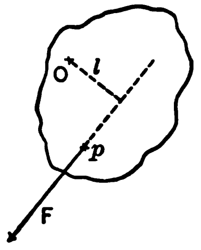

Fig. 42.

67. Components of torque about the axes of reference. — Consider a force of which the x, y, and z components are X, Y, and Z. Let x, y, and z be the coördinates of the point of application of this force; then the torque actions of this force about the x, y, and z axes are, respectively

$$\begin{aligned} T_x &= Zy - Yz \\ T_y &= Xz - Zx \\ T_z &= Yx - Xy \end{aligned} \qquad (38)$$

Proof. — Consider only T_z. From Fig. 43 it is evident that Yx is the torque action of Y about the z-axis, and that $-Xy$ is the torque action of X about the z-axis, this latter being considered negative because it tends clockwise while Yx tends counter-clockwise. Therefore $Yx - Xy$ is the total torque action of the given force about the z-axis, inasmuch as the component Z has no tendency to turn the body about the z-axis.

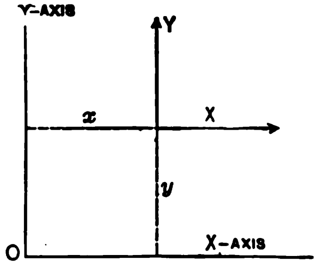

Fig. 43.

Origin of moments. — The point about which the torque action of a number of forces is reckoned is called the origin of moments.

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* Sometimes called the *moment of a force.* The discussion of rotatory motion is given in Chapter V.; however, some statements concerning angular velocity, torque, etc., are given in the present chapter.

68. Second condition of equilibrium. — In order that a number of forces may have no tendency to turn a body upon which they act, it is *necessary*[*] that the sum of the torque actions of the forces about each axis of reference be zero. That is,

$$\begin{aligned} \Sigma\,(Zy - Yz) &= 0 \\ \Sigma\,(Xz - Zx) &= 0 \\ \Sigma\,(Yx - Xy) &= 0 \end{aligned} \qquad (39)$$

69. Couple. — A number of forces may satisfy the first condition of equilibrium without satisfying the second condition. In this case the forces tend to produce rotation only, and they constitute a torque. Two equal and opposite forces which do not satisfy the second condition of equilibrium (that is, the two equal and opposite forces do not act in the same line) are called a *couple.*

70. Three forces in equilibrium intersect at a point. — The forces must lie in one plane in order to satisfy the first condition of equilibrium. Choose the origin of moments at the intersection of two of the forces. Then the third force must pass through this point, otherwise it will have an unbalanced torque action about the point.

71. Any number of forces, not in equilibrium, acting on a body are together equivalent to a single force, which is called their resultant; except when the forces constitute a couple.

Proof. — Given a number of forces in equilibrium, omitting *one* of these forces, the combined action of the others must be equivalent to an equal and opposite force having the same line of action. The exception is also evident, since by omitting one of a set of forces in equilibrium the others cannot constitute a couple. The magnitude and direction of the resultant of a number of forces is determined as their vector sum. The point of application of this resultant is the point about which the given forces have no torque action.

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[*] It can be shown that this is also a sufficient condition provided the first condition of equilibrium is satisfied.

72. The center of mass of a body is the point of application of the resultant of the parallel forces with which gravity acts upon the particles of the body.

Proof. — Let the origin of coördinates O, Fig. 44, be chosen at the center of mass of the body, let the X-axis be downwards. The force acting on a particle of mass Δm is $\Delta m \cdot g$, g being the acceleration of gravity. The torque action of this force about the z-axis is $gy\Delta m$, and the total torque action of the forces on all the particles is $g\Sigma y\Delta m$, which is zero by equation (20). That is the total torque action of the forces $\Delta m \cdot g$ is zero about the center of mass of the body, which is therefore the point of application of the resultant of the forces $\Delta m \cdot g$.

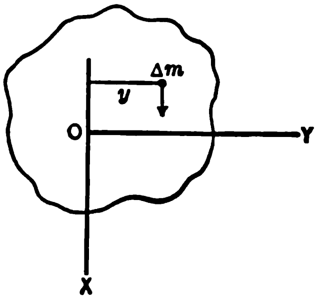

Fig. 44.

Work and Energy.

73. Work and energy. — Whenever the point of application of a force moves in the direction of the force, the force is said to do *work*. The work W done by a force F in pulling a body through a distance d is

$$W = Fd \tag{40}$$

If F and d are not parallel, then $W = Fd\cos\theta$, where θ is the angle between F and d.

74. Units of work or energy. — The work done by unit force in pulling a body unit distance is the unit of work or energy.

The work done by one dyne in pulling a body a distance of one centimeter is called the *erg*. This unit of work is, for most purposes, inconveniently small. The *joule* is equal to 10^7 ergs.

The work done by a force of one kilogram's weight pulling a body through a distance of one meter is called the *kilogram-meter*.* This unit of work is much used by engineers.

Examples. — A horse pulls a plow with a force of 100,000,000 dynes (about 100 kilograms' weight) through a distance of 1000 cm. The work done, viz., 1×10^{11} ergs or 1×10^4 joules, is all dissipated on account of friction.

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* English engineers use the *foot-pound* as the unit of work.

A locomotive pulls with a force of 1×10^{10} dynes (about 10,000 kilograms' weight) through a distance of 180,000 cm. in bringing a train to full speed. The work done, viz., 18×10^{14} ergs, or 18×10^{7} joules, is (ignoring friction), stored in the train as kinetic energy. (See Art. 79.)

The hoisting rope of a mine pulls with a force of 25×10^{8} dynes (about 2500 kilograms' weight) upon the cage. The work done by this force in lifting the cage through a height of 10,000 cm., viz., 25×10^{12} ergs, or 25×10^{5} joules, is stored in the cage as potential energy. (See Art. 80.)

75. Power. — The rate at which an agent does work is called the *power* of that agent.

Units of power. — The c.g.s. unit of power is *one erg per second.* The *watt* is *one joule per second*, or 10^7 *ergs per second.*

The *horse-power* * is 746 watts.

76. Power developed by a force acting on a moving body. — Consider a force F, acting upon a body which is moving at velocity v in the direction of F. During the interval Δt the body moves a distance $v \cdot \Delta t$, and the work done during this interval by the force F is $\Delta W = Fv \cdot \Delta t$, which divided by the interval Δt, gives the rate at which work is done, or the power, P, developed by the force; that is

$$P = Fv \tag{41}$$

77. Work done by a torque, and power developed by a torque. — The work W done by a torque T in turning a body through an angle of ϕ radians (about the axis of the torque), is

$$W = T\phi \tag{42}$$

Proof. — Consider a torque T due to a force F, Fig. 45, acting as shown. Then $T = Fl$. When the wheel has turned through an angle ϕ, the point of application of the force will have moved a distance ϕl in the direction of the force, and an amount of work $W = F\phi l$, or $W = T\phi$, will have been done.

Q.E.D.

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* The horse-power is 550 foot-pounds per second.

The power P developed by a torque T, which turns a body at angular velocity ω, is

$$P = T\omega \tag{43}$$

Proof. — In time Δt the body will turn through the angle $\omega \cdot \Delta t$, and an amount of work $T\omega \cdot \Delta t$ will be done; this work divided by Δt gives the rate at which work is done, or the power developed, so that $P = T\omega$. Q.E.D.

Examples. — A belt pulls with a net* force of 5×10^7 dynes (about 50 kilograms' weight) on the rim of a pulley, the velocity of the rim being $1000 \frac{\text{cm.}}{\text{sec.}}$ The power transmitted by the belt is 5×10^{10} ergs per second, 5×10^3 watts, or nearly 7 horse-power.

A shaft subjected to a torque of 5×10^8 dyne-cm. (equivalent to about 50 kilograms' weight on a 10 cm. arm) runs at a speed of 5 revolutions per second, or 10π radians per second. The power transmitted along the shaft is $5\pi \times 10^9$ ergs per second, 500π watts, or about 2 horse-power.

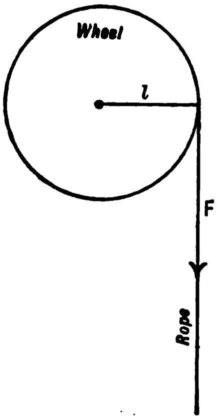

Fig. 45.

78. Measurement of power. — The power of an agent may be measured in the following ways:†

(*a*) By measuring a force and the velocity of the body upon which it acts. Equation (41) is then used to calculate P.

(*b*) By measuring a torque and the angular velocity of the body which it turns. Equation (43) is then used to calculate P.

Example. — Let a be the area of the piston of a steam engine and p the average steam pressure in the cylinder. Then the average force pushing on the piston is $F = ap$. Let l be the length of stroke of engine and n its revolutions per second.

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[*] The difference in tension on the tight and loose sides of the belt.

[†] In an electric circuit (see Vol. II., Art. 97) the power is determined by measuring an electromotive force and the electric current which it produces. The power developed is equal to the product of the two.

Then the average velocity of the piston is $v = 2ln$, and the power developed is $P = Fv = 2apln$. The quantities a, l, and n are easily determined. The average steam pressure is determined by means of an instrument called the *steam-engine indicator*.

79. Energy of a system of particles. — When a system of particles by virtue of its *state of motion** is capable of doing work, it is said to possess or store energy. Its energy is measured by the work which it can do.

Kinetic Energy. — The energy stored in a system by virtue of its *velocity configuration** is called its *kinetic energy*. For example, a moving train possesses energy because of its motion.

Proposition. — The kinetic energy of a moving particle is

$$W = \tfrac{1}{2}mv^2 \tag{44}$$

in which m is the mass of the particle and v is its velocity.

Proof. — The kinetic energy of a particle is measured not only by the work which it can do when stopped, but also by the work required to establish its motion. Let a constant force F act upon a particle of mass m, starting it from rest. This force imparts velocity to the particle at a rate $\dot{v}$ such that

$$F = m\dot{v} \tag{i}$$

After a time t has elapsed, the velocity gained by the particle is

$$v = \dot{v}t \tag{ii}$$

and the distance traveled by the particle (equation (12)) is

$$d = \tfrac{1}{2}\dot{v}t^2 \tag{iii}$$

Multiplying equations (i) and (iii) member by member, and remembering that Fd is the work W which has been done, we have

$$W = \tfrac{1}{2}m\dot{v}^2t^2 \tag{iv}$$

But

$$\dot{v}^2t^2 = v^2 \qquad \text{by equation (ii)}$$

so that

$$W = \tfrac{1}{2}mv^2 \qquad \text{Q.E.D.}$$

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* See Art. 49.

The kinetic energy of a system of particles is the sum of the kinetic energies of the individual particles. That is,

$$W = \tfrac{1}{2}\Sigma mv^2 \tag{45}$$

80. Potential Energy. — The energy stored in a system by virtue of its position configuration * is called the *potential energy* of the system. For example, a weight stores energy by virtue of its position relative to the earth; a bent spring stores energy by virtue of its configuration.

It is necessary to adopt arbitrarily some certain configuration of a system, called the *zero configuration*, for which the potential energy is an unknown constant, and to reckon the potential energy of any other given configuration as the work necessary to change the system from the zero configuration to the given configuration. It is pointed out in Art. 83 that the work required to change a system from one configuration to another is the same whatever the intermediate stages may be through which the system is made to pass.

81. Dissipation of Energy. — In every actual case of motion the moving bodies are subject to friction and to collision. The moving bodies are thus brought to rest and their energy is said to be *dissipated*. The principles discussed in the following paragraphs have immediate reference to *ideal systems in which energy is not dissipated*. In the chapters on heat these principles are extended to include systems in which energy is dissipated.

82. Mutual relation between the kinetic energy and potential energy of a closed † system. — Consider such a movement of the particles of a closed system, that the forces with which the particles act on each other do work. The kinetic energy of the system increases by an amount equal to the work done, and the potential energy of the system decreases by the same amount. Therefore *the sum of the kinetic energy and potential energy of a closed system does not change.*

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* See Art. 49.

† A closed system is one upon which no forces act from the outside.

83. Principle of the conservation of energy. Completion of the definition of potential energy. — *The work required to change a system from one configuration to another is independent of the intermediate stages through which the system is made to pass.* If this were not true, the potential energy of a system in a given configuration could not be defined as the work required to bring the system from the zero configuration to the given configuration, for this work would not be definite in amount. Having assurance that the conception of potential energy is legitimate, the statement given in Art. 82 is legitimate also, so that

The total energy of a closed system is constant.

If the total energy of one system (not closed) changes, the energy of some other system must change by an equal and opposite amount, for the two systems may be looked upon as one closed system. Therefore

Energy cannot be created or destroyed.

Remark. — The principle of the conservation of energy (second, third, and fourth lines of this article) is a result of experience, and the student must not expect a proof of it.

84. Example of a system illustrating the definiteness of potential energy. — Consider a body attracted by the earth. The work required to lift this body from a point A to a point B (ignoring the energy which is dissipated by friction) is the same whatever the path may be over which the body is carried. Let d be the vertical distance between the two points A and B. The work done in lifting the body a distance d is mgd, where mg is the weight of the body. Therefore the potential energy of the body at the upper point exceeds its potential energy at the lower point by the amount mgd.

85. Example of motion illustrating the constancy of the total energy of a closed system. — Consider a projectile of mass m thrown from O (Fig. 46) with initial velocity v_1 in the direction OP. The kinetic energy of the projectile is $\frac{1}{2}mv_1^2$.

After t seconds the projectile will have gained vertical velocity of amount gt, so that the velocity of the projectile after t seconds will be represented by OP', and its square is equal to $v_1^2 + g^2t^2 - 2v_1gt\sin\theta$. Therefore the kinetic energy of the projectile after t seconds is

$$\tfrac{1}{2}m(v_1^2 + g^2t^2 - 2v_1gt\sin\theta)$$

and the *increment* of kinetic energy is

$$\tfrac{1}{2}mg^2t^2 - mv_1gt\sin\theta \qquad \text{(i)}$$

The distance of the projectile below O (Fig. 47) after t seconds is $d = \frac{1}{2}gt^2 - v_1t\sin\theta$, and the *decrement* of the potential energy of the projectile (with respect to the earth) is equal to the product of its weight mg into d; namely

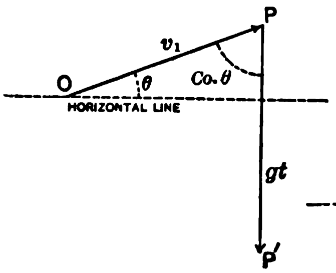

Fig. 46.

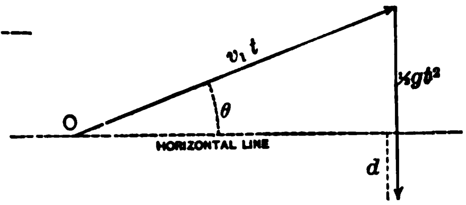

Fig. 47.

$$\tfrac{1}{2}mg^2t^2 - mv_1gt\sin\theta \qquad \text{(ii)}$$

The identity of (i) and (ii) shows that the increment of kinetic energy is equal to the decrement of potential energy. If the resistance of the air is perceptible, then the projectile continually gives up energy to the air, and the projectile and earth do not constitute a closed system.

CHAPTER V.

ROTATORY MOTION.

86. Rigid body; angular velocity; angular acceleration. — A rigid body is a body so stiff that there can be no perceptible relative motion of its parts. A rigid body can perform only translatory and rotatory motion. (Compare Art. 32.)

The average *angular velocity* of a rigid body during an interval of time Δt is the quotient $\Delta\phi/\Delta t$ where $\Delta\phi$ is the angle turned by the body about its axis of rotation in the interval. When the angular velocity of a body is changing, its value at a given instant is the value of the quotient $\Delta\phi/\Delta t$ when the interval Δt is very short. (Compare Art. 26.)

Angular velocity is usually expressed in *radians per second* and it is represented by the letter ω; it may also be expressed in *revolutions per second*, n. There are 2π radians in a revolution and therefore there are $2\pi n$ radians per second (ω) in n revolutions per second; that is

$$\omega = 2\pi n \tag{46}$$

Graphical representation of angular velocity, angular acceleration, and torque. An angular velocity is a vector quantity and it may be represented by a line. The line OP, Fig. 48, which represents a given angular velocity ω is drawn *parallel to the axis of rotation of the body* and of such length as to represent the radians per second (ω) to a convenient scale. The positive direction of the line, as indicated by the arrow-head, is chosen in that direction along the axis in which the body would move if it were a nut turning on a right-handed screw.

If an unbalanced turning force or torque acts upon a rotating body the angular velocity* of the body will change either by making the angular velocity larger or smaller, or by changing the direction of the axis of rotation, or both. The rate $(\Delta\omega/\Delta t)$ at which the angular velocity of a body changes is called the *angular acceleration* of the body.

Let the line OP, Fig. 48, represents the angular velocity of a body at a given instant; let the line OP' represent the changed angular velocity after the lapse of the time interval Δt; and let the line PP' represent the change $\Delta\omega$ of the angular velocity during the interval Δt.

The meaning of the diagram, Fig. 48, is as follows: at the given instant the body is rotating about OP as an axis; an external agent acts upon the body tending to produce rotation about PP' as an axis, and the body then rotates about OP' as an axis; the original radians per second are represented by the length of OP, the radians per second produced by the external agent are represented by the length of PP', and the final radians per second are represented by the length of OP'. This vector addition of angular velocities is beautifully illustrated by the gyroscope. (See Art. 99.)

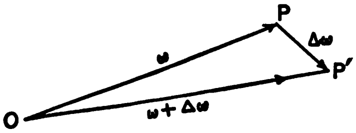

Fig. 48.

87. Motion of a rigid body in a plane. — A rigid body is said to move in a plane when all points of the body which lie in the plane remain in it. For example, a rotating wheel moves in a plane, the connecting rod of a steam engine moves in a plane. Consider a rigid body AB, Fig. 49, moving in the plane of the paper. The position of the body is completely indicated by the position of the line AB fixed in the body. This line is called the **index line.**

After any change in the position of a rigid body moving in a plane, a certain line in the body perpendicular to the plane is in its initial position, and the given displacement is equivalent to a rotation about that line as an axis. Let AB and $A'B'$, Fig. 50, be the positions of the index line before and after the displacement. Join AA' and BB'. Erect perpendiculars from the middle points of AA' and BB' intersecting at p. From the similarity of the triangles pAB and $pA'B'$ it is evident that the same part of the body is at p before and after the displacement, and that the line through p perpendicular to the paper is the line about which the body may, by simple rotation, move from its initial to its final position. The angle $\Delta\phi$ of this rotation is the angle subtended by AA' or BB' as seen from p.

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* The angular velocity of a non-symmetrical rotating body changes continuously even when the body is entirely free from force action from the outside.

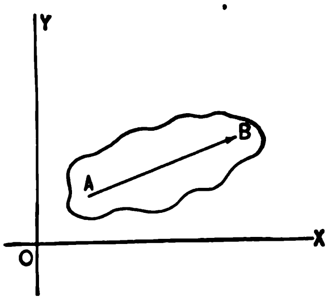

Fig. 49.

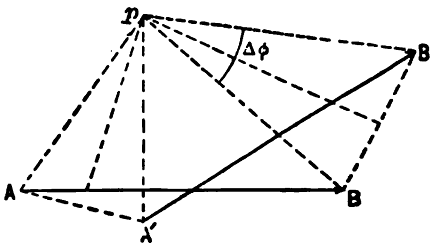

Fig. 50.

88. The instantaneous motion of a rigid body moving in a plane in any manner, is a motion of rotation about a definite line called the **instantaneous axis** of the motion. Let the displacement shown in Fig. 50 be that which takes place in a short interval of time Δt; then $\Delta\phi/\Delta t$ is the instantaneous angular velocity of the body, and the line through p, perpendicular to the paper, is the instantaneous axis. During a finite interval of time the motion of a body may be irregular. Forces, however, never being infinite, the motion of a body during an interval approaches uniformity as that interval approaches zero. Therefore the motion of a body during a short interval of time is the simplest motion which can produce the actual displacement which occurs during that interval.

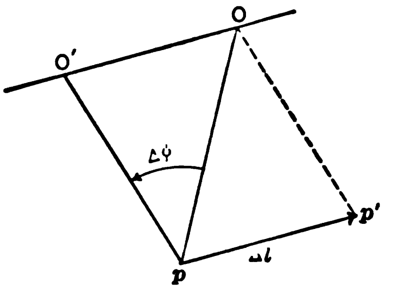

Fig. 51.

89. Composition of angular and linear displacements. — Consider an angular displacement $\Delta\phi$ of a body about the point p, Fig. 51, bringing the point O to O'; and a linear displacement Δl parallel and equal to

$O'O$, bringing O' back to O. These two displacements are equivalent to an angular displacement $\Delta\phi$ about O, bringing Op to Op'. Let the distance of p from the line OO' be r; then if $\Delta\phi$ is small, $\Delta l = r\Delta\phi$.

90. Resolution of motion in a plane. — From Arts. 88 and 89 it follows that the instantaneous motion of a rigid body in a plane may be resolved into a motion of rotation about an arbitrary point combined with a linear velocity. Consider the actual displacement represented in Fig. 51, namely a rotation about O bringing p to p'. This displacement is equivalent to an equal angular displacement $\Delta\phi$, about the arbitrary point p, together with the linear displacement $O'O$ or pp'. Let this linear displacement be Δx, and let Δt be the interval which elapses during the displacement. The actual angular velocity $\Delta\phi/\Delta t$ about the point O (the instantaneous center) is equivalent to an angular velocity $\Delta\phi/\Delta t$ about the point p combined with a linear velocity $\Delta x/\Delta t$ parallel to pp'.

91. Motion of a rigid body with one point fixed. *Proposition.* — If a rigid body, one point of which is fixed, is displaced in any manner whatever, a certain line in the body will be in its initial position after the displacement, and the given displacement will be equivalent to a rotary movement about this line as an axis.

Proof. — Consider a spherical shell of the body having its center at the fixed point. Let AB, Fig. 52, be an arc of a great circle on this spherical shell; the position of AB fixes the position of the body, and AB is called the *index line*. Let the movement of the body bring AB to $A'B'$. Connect AA' and BB' by arcs of great circles. Draw great circles bisecting AA' and BB' at right angles. The point p at the intersection of these circles bisecting AA' and BB' has the same position relative to AB and $A'B'$, so that this point of the shell is in its initial position, and the line drawn from the center of the spherical shell to the point p is the axis about which the given movement can be produced by rotation.

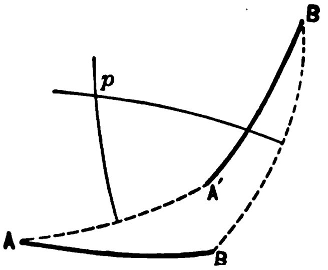

Fig. 52.

Corollary. — The instantaneous motion of a rigid body about a fixed point is a motion of simple rotation at definite angular velocity about a definite line called the instantaneous axis of the motion.

92. Kinetic energy of a rotating rigid body; definition of moment of inertia.—The kinetic energy of a given rigid body rotating about a fixed axis is proportional to the square of its angular velocity. This is evident when we consider that the *linear velocity* of every particle in the body would be doubled if we were to double the angular velocity of the body, or the kinetic energy of each and every particle of the body would be quadrupled; so that the total kinetic energy of the body would be quadrupled by doubling its angular velocity. We may therefore write, for a given body rotating about a given axis:

$$W = \tfrac{1}{2}K\omega^2 \qquad (47)$$

in which $\tfrac{1}{2}K$ is the proportionality factor. The quantity K is called the *moment of inertia* of the given body about the given axis. The origin of the term *moment of inertia* is explained in Art. 96.

Remark. — The definition of moment of inertia here given and the discussion given in the following articles, 93 to 99, apply only to a rotating body which is entirely symmetrical with respect to its axis of rotation. The axis of rotation of such a body remains stationary when the body is not acted upon by outside forces, and if the body is acted upon by outside force as described in Art. 99, the axis of rotation will move as there specified.

The rotatory motion of non-symmetrical bodies cannot be discussed in this text.

93. Dependence of moment of inertia upon mass and dimensions of a body. — Consider a rigid body, Fig. 53, rotating at angular velocity ω about the axis p, which is perpendicular to the paper. The linear velocity v of the particle of mass Δm, distant r from p, is $v = r\omega$, and the kinetic energy of this particle (equation 44) is $\tfrac{1}{2}\Delta mr^2\omega^2$. The total kinetic energy of the rotating body therefore is:

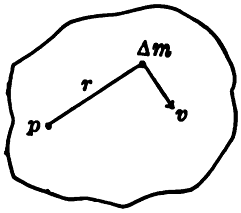

Fig. 53.

$$W = \Sigma\tfrac{1}{2}\Delta mr^2\omega^2$$

or

$$W = \tfrac{1}{2}\omega^2\Sigma r^2\Delta m \qquad (48)$$

Comparing equation (48) with equation (47), we have

$$K = \Sigma r^2\Delta m \qquad (49)$$

That is, the mass of each particle of the body is to be multiplied by the square of its distance from the axis, and the sum of such products taken.

Example of the calculation of moment of inertia. — Consider a circular disk of mass M and radius R rotating about its axis of figure. The mass per unit area of face of disk is $M/\pi R^2$. The mass, Δm, of the portion of the disk which lies between the circles of radii r and $r + \Delta r$ is:

$$\Delta m = \frac{M}{\pi R^2} \cdot 2\pi r \Delta r$$

Substituting this value of Δm in equation (49), we have:

$$K = \Sigma \frac{2M}{R^2} \cdot r^3 \Delta r = \frac{2M}{R^2} \Sigma r^3 \Delta r$$

The sum $\Sigma r^3 \Delta r$, taken between the limits $r = 0$ and $r = R$, is equal to $\frac{1}{4} R^4$, so that

$$K = \tfrac{1}{2} M R^2$$

That is, the moment of inertia of the disk is equal to one half the product of its mass into the square of its radius, and the radius of gyration of the disk is the length whose square is equal to $\frac{1}{2} R^2$; that is, $R/\sqrt{2}$.

MOMENTS OF INERTIA OF SOME REGULAR HOMOGENEOUS SOLIDS (AXIS OF ROTATION PASSING THROUGH CENTER OF MASS).

Body.	Value of K.
Sphere of radius r	$\frac{2}{5} mr^2$
Cylinder of radius r, axis of cylinder is the axis of rotation	$\frac{1}{2} mr^2$
Slim rod of length l, axis of rotation at right angles to rod.	$\frac{1}{12} ml^2$
Rectangular parallelepiped of length l and breadth b axis of rotation at right angles to l and to b	$\frac{1}{12} m(l^2+b^2)$

94. Radius of gyration. — The radius of gyration of a body rotating about a fixed axis is the distance ρ from the axis, at which the entire mass m of the body might be concentrated without altering the moment of inertia of the body. If the entire mass of a body were concentrated at distance ρ from the axis the moment of inertia of the body would be equal to mp^2 according to equation (49).

That is

$$K = m\rho^2 \tag{50}$$

or

$$\rho = \sqrt{\frac{K}{m}} \tag{51}$$

Thus the radius of gyration of a sphere is $\sqrt{\frac{2}{5}}$ times the radius of the sphere, the radius of gyration of a cylinder rotating about its axis of figure is $\sqrt{\frac{1}{2}}$ times the radius of the cylinder.

Imagine a rotating body to be divided up into a very large number n of particles of equal mass Δm. Then $\Delta m = m/n$. Substituting this value of Δm in equation (49) we have

$$K = m \times 1/n\, \Sigma r^2$$

Comparing this with equation (50) we have

$$\rho^2 = 1/n\, \Sigma r^2$$

Now Σr^2 is the sum of the squares of the distances of all the particles (of equal mass) of the body from the axis, and $1/n\, \Sigma r^2$ is the average value of the squares of the distances of the particles (of equal mass) from the axis. Therefore the radius of gyration of a body is equal to the square-root-of-the-average-square of the distances of the various particles (of equal mass) of the body from the axis.

95. Relation between moments of inertia about parallel axes. — Let K be the moment of inertia of a body of mass m about a given axis passing through the center of mass of the body, and K' its moment of inertia about another axis parallel thereto and distant a from the center of mass; then

$$K' = K + a^2 m \tag{52}$$

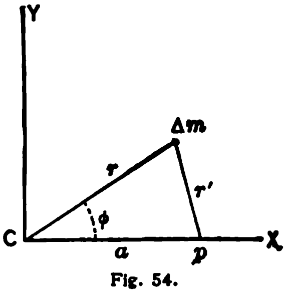

Fig. 54.

Proof. — Let C, Fig. 54, be the center of mass of the body, chosen as the origin of coördinates, and let p be the axis, perpendicular to the plane of the paper, about which K' is reckoned. Consider a particle of mass Δm, distant r from C, distant r' from p, and of which the coördinates are x and y.

From trigonometry we have from the triangle $C\,p\,\Delta m$, Fig. 54:

$$r'^2 = r^2 + a^2 - 2ra\cos\phi \tag{i}$$

From equation (49) we have

$$K' = \Sigma r'^2\Delta m \tag{ii}$$

Substituting the value r' from (i) in (ii) we have

$$K' = \Sigma r^2\Delta m + a^2\Sigma\Delta m - 2a\Sigma r\cos\phi\,\Delta m \tag{iii}$$

But $\Sigma r^2\Delta m = K$, from equation (49), and $a^2\Sigma\Delta m = a^2m$. Furthermore, $\Sigma r\cos\phi\,\Delta m = \Sigma x\Delta m$, which is equal to zero from equation (20), inasmuch as the origin of coördinates is the center of mass of the body. Therefore equation (iii) reduces to

$$K' = K + a^2m$$ Q.E.D.

96. Relation between torque and angular acceleration. — A torque T acting on a body of which the moment of inertia is K, imparts angular velocity to the body at the rate $\dot{\omega}$ such that

$$T = K\dot{\omega} \tag{53}$$

The idea of moment of inertia first arose in connection with this matter of torque and angular acceleration. Consider, for example, a 10-kilogram ball fixed to one end of a 2-meter arm which is pivoted at the other end so as to be free to rotate like the spoke of a wheel. On account of the inertia of the 10-kilogram ball a considerable turning force or torque would have to act upon the arm to impart angular velocity to it. The reluctance of the ball and arm to gain angular velocity about the pivot is called the moment of inertia of the ball with respect to the pivot, and, according to equation (49), Art. 93, the moment of inertia of the ball with respect to the pivot is equal to the product of the mass of the ball into the square of the length of the arm.

Proof of equation (53). — From equation (47) we have $dW/dt = K\omega\, d\omega/dt$, or, since the rate of increase of energy of a rotating body is equal to the power P expended on the body, we have $P = K\omega\dot{\omega}$, where $\dot{\omega}$ is written for $d\omega/dt$. Now the power expended on a rotating body is always equal to the product of the driving torque T into the angular velocity ω (equation 43). Therefore $P = K\omega\dot{\omega} = T\omega$, or, $T = K\dot{\omega}$. Q.E.D.

97. Correspondence between translatory motion and rotatory motion. — To every equation in translatory motion there is a corresponding, similar equation in rotatory motion in which moment of inertia takes the place of mass, angle takes the place of distance, angular velocity takes the place of linear velocity, angular acceleration takes the place of linear acceleration, torque takes the place of force, etc. The following table exhibits the pairs of equations which have been already discussed.

TABLE.

TRANSLATORY MOTION.		ROTATORY MOTION.	
$F = m\dot{v}$	(8)	$T = K\dot{\omega}$	(53)
$W = Fd$	(40)	$W = T\phi$	(42)
$P = Fv$	(41)	$P = T\omega$	(43)
$W = \frac{1}{2}mv^2$	(44)	$W = \frac{1}{2}K\omega^2$	(47)

In addition to these there are equations of harmonic rotatory motion corresponding to the equations of harmonic translatory motion; namely,

TRANSLATORY MOTION.		ROTATORY MOTION.	
$F = -ax$	(33)	$T = -b\phi$	(54)
$a = 4\pi^2n^2m$	(34)	$b = 4\pi^2n^2K$	(55)

The meaning of equations (54) and (55) is explained in the next article.

98. Proposition concerning harmonic rotatory motion. — From equations (54) and (55) we have the following proposition corresponding to the proposition given in Art. 61.

If a body is held so as to be urged back towards its equilibrium position by a torque

$$T = -b\phi \qquad (54)$$

in which ϕ is the angle through which the body has been turned from its equilibrium position and b is a constant; then if started, the body will perform simple harmonic rotatory motion of such frequency n that

$$b = 4\pi^2n^2K \qquad (55)$$

in which K is the moment of inertia of the body. If we substitute for n the value $1/\tau$, where τ is the period of one vibration, equation (55) becomes

$$b = \frac{4\pi^2 K}{\tau^2} \tag{56}$$

An example of harmonic rotatory motion is discussed in the article on the Torsion Pendulum.

99. The gyroscope is a wheel mounted on an axle which is entirely free to change its direction. When the wheel is set rotating rapidly the direction of the axle remains unaltered so long as no external force acts upon it. This permanency of direction of axis of rotation of a symmetrical rotating body when no external force acts upon it is exemplified approximately by the earth, the axis of rotation of which is nearly * invariable in direction.

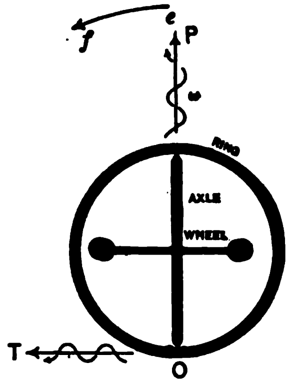

Fig. 55.

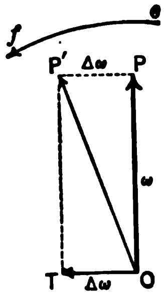

Fig. 56.

The behavior of a rotating symmetrical body under the action of outside force may be illustrated by means of the gyroscope as follows: Figure 55 shows a *top view* of a gyroscope. The metal ring rests upon a pivot underneath O. When the gyroscope is supported in this manner the pull of the earth on the ring and wheel produces a torque about the line T, Fig. 55, as an axis.

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* The attraction of the sun and moon upon the bulging equatorial portion of the earth's mass causes the axis of the earth to turn slowly about a line perpendicular to the plane of the earth's orbit, just as the axis of a top turns slowly about a vertical line.

The effect of this torque is to make the axis of the gyroscope sweep round the point O in the direction of the arrow ef. Some insight into this action of the gyroscope may be obtained with the help of the geometrical construction shown in Fig. 56, the lettering of which corresponds to the lettering of Fig. 55. The line OP parallel to the axis of the wheel represents the angular velocity ω of the wheel at a given instant. The effect of the torque T during a small interval of time Δt is to impart to the gyroscope an angular velocity $\Delta\omega$ $(= T.\Delta t/K)$ about the axis OT. This angular velocity is represented by the line OT, Fig. 56, and it is added vectorially to the initial angular velocity ω giving a resultant angular velocity about the line OP' *which is the position of the axle after lapse of time* Δt.

The action of the gyroscope is of course to be explained fundamentally by considering the motion of the various particles of the wheel. Consider a circular disk, Fig. 57, rotating rapidly in the direction ef, at an angular velocity ω, about an axis O perpendicular to the plane of the disk. Suppose this *axis* to turn slowly at an angular velocity ω', about the fixed line AB, in such direction as to bring the limb C of the disk towards the reader. Consider the velocity v of the successive particles of the disk as they pass the point p. If the axis O were fixed this velocity v would be constant both in magnitude and in direction, but since the axis O is supposed to be turning about AB the velocity v is continually changing its direction. From a point O' draw a line $O'P'$ representing v, then as the axis O turns round AB the line $O'P'$, which must be at each instant parallel to v, will sweep round a circular cone at angular velocity ω' and at the instant when the axis O is perpendicular to the plane of the paper the velocity of the point P' is perpendicular to the plane of the

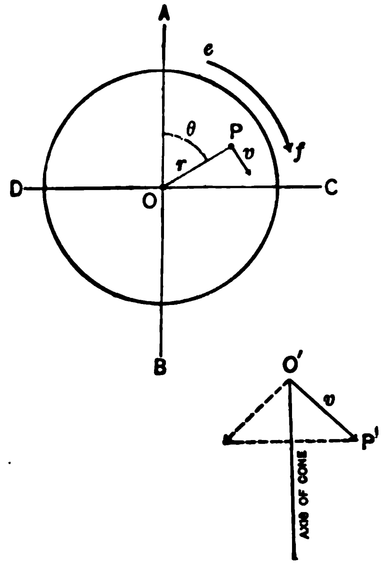

Fig. 57.

paper and this velocity of P' represents $\dot{v}$ to the same scale that $O'P'$ represents v (see Art. 35). In this way it may be easily shown that $\dot{v}$ is *towards the reader* for every point p of the disk which is above the line DC, and *away from the reader* for every point p below the line DC. Therefore external forces must be pushing every particle of the disk above DC *towards the reader*, and external forces must be pushing every particle of the disk below DC *away from the reader* in order that the axis O may move in the prescribed manner. But the resultant of all these external orces is an external torque about the axis DC. Therefore a steady external unbalanced torque about the diameter DC of the disk will cause the axis O of the rotating disk to sweep around the line AB.

100. The torsion pendulum. — A body hung by a wire is in equilibrium when the wire is untwisted. When turned about the wire as an axis through an angle ϕ from its equilibrium position, the body is urged back by the wire by a torque T which is proportional to ϕ, so that

$$T = -b\phi \qquad (54) \text{ bis}$$

The proportionality factor b is called the *constant of torsion* of the wire. If the body be started, it will perform harmonic rotatory motion of such frequency that

$$\frac{4\pi^2 K}{\tau^2} = b \qquad (56) \text{ bis}$$

in which K is the moment of inertia of the body and τ is the period of its vibrations. A body vibrating in the manner here described is called a *torsion pendulum*. The balance wheel of a watch is a torsion pendulum, the governing torque being due to the hair spring.

Remark. — The constant of torsion of a wire may be calculated from equation (56), when K is known and τ observed.

101. Use of torsion pendulum in the comparison of moments of inertia. — If two bodies are hung from the same wire in succession and their periods of torsional vibration τ and τ' observed, we have

$$\frac{4\pi^2 K}{\tau^2} = b$$

$$\frac{4\pi^2 K'}{\tau'^2} = b$$

whence
$$\frac{K}{K'} = \frac{\tau^2}{\tau'^2} \qquad (57)$$

in which K and K' are the respective moments of inertia of the bodies.

102. Use of torsion pendulum in the measurement of very small forces. — When used for this purpose, the torsion pendulum is called a *torsion balance*. With such an instrument, the essential features of which are shown in Fig. 58, Coulomb investigated the laws of electrostatic attraction and repulsion, and the mutual action of magnets. The torsion pendulum is used as follows: A body of known moment of inertia is suspended by the wire, and the period of its vibrations is observed. The constant of torsion b of the wire is then calculated from equation (56). The wire is then used to suspend two bodies B and C, Fig. 59, which are

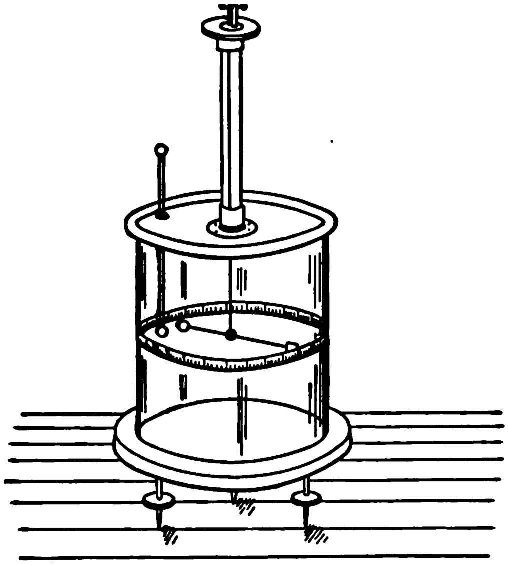

Fig. 58.

fixed to the ends of a horizontal arm AA, the wire being attached at the point W. The small force F to be measured is then made to act horizontally upon the body B. The effect of this force is to twist the suspending wire through an observed angle ϕ, and since the torque action Fl of the force is balanced by the reacting torque $b\phi$ of the twisted wire we have

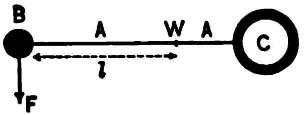

Fig. 59.

$$Fl = b\phi$$

or

$$F = \frac{b\phi}{l}$$

103. Cavendish's experiment* for the determination of the constant of gravitation (see Arts. 54 and 55). — Two lead balls, BB' (Fig. 60), carried by a light horizontal bar ll, were suspended by a wire from a point W (the plane of the paper is horizontal), forming a torsion balance. Two massive lead balls, S, S' were arranged as shown, and the gravitational attraction between S and B, and S' and B' was measured as explained above. From this measurement Cavendish calculated the constant of gravitation and the mass of the earth as explained in Arts. 54 and 55.

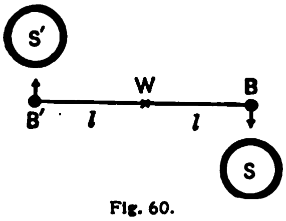

Fig. 60.

104. The gravity pendulum consists of a rigid body AB, Fig. 61, suspended so as to be free to turn about a horizontal axis p.

Let C, Fig. 61, be the center of mass of the body AB. This point C is vertically below p when the body is in equilibrium. Let the body be turned to one side through the angle ϕ as shown. Then the force, mg, with which the earth pulls the body AB tends to turn the body *back* into the vertical position with a torque T which is equal to the product of mg into the distance aC'. But the distance aC' is equal to $x \sin \phi$ where x is the distance pC. Therefore

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* Repeated by Baille (1842), Reich (1852), Cornu and Baille (1878).

$$T = -mgx \sin \phi \qquad \text{(i)}$$

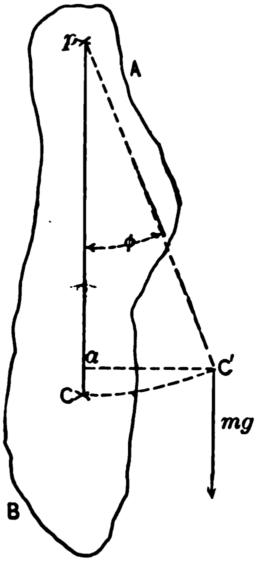

Fig. 61.

When ϕ is small then $\sin \phi = \phi$, very nearly, and equation (i) becomes

$$T = -mgx \cdot \phi \qquad \text{(ii)}$$

Comparing this with equations (54) and (56), Art. 98, we find that

$$\frac{4\pi^2 K}{\tau^2} = mgx \qquad (58)$$

in which K is the moment of inertia of the body about the axis p, τ is the period of one complete pendulous vibration of the body, and g is the acceleration of gravity.

Remark. — A pendulum such as is described in this article is sometimes called a *physical pendulum* to avoid confusion with the ideal *simple pendulum* described in the next article

105. The simple pendulum. Equivalent length of a physical pendulum. An ideal pendulum consisting of a particle of mass m suspended by a weightless cord or rod of length l is called a *simple pendulum.* The moment of inertia of such a pendulum about the supporting axis p is $K = ml^2$ according to equation (49). Furthermore the center of mass of a simple pendulum is, of course, at the center of the suspended particle. Therefore, for the simple pendulum, we may write ml^2 for K, and l for x in equation (58), whence we have

$$\frac{4\pi^2 l}{\tau^2} = g \qquad (59)a$$

or

$$l = \frac{\tau^2 g}{4\pi^2} \qquad (59)b$$

in which l is the length of a simple pendulum, τ is the period of one complete vibration, and g is the acceleration of gravity.

Equivalent length of a physical pendulum. — The quantity $\tau^2 g/4\pi^2$, in which τ is the period of vibration of any given physical pendulum, is the length of a simple pendulum having the same period, and this length is called the equivalent length of the given physical pendulum. Solving equation (58) for $\tau^2 g/4\pi^2 (= l)$ we have:

$$l = \frac{K}{mx} \tag{60}$$

in which l is the equivalent length of a pendulum, K is the moment of inertia of the pendulum about its axis of support, m is the mass of the pendulum, and x is the distance from the axis of support to the center of mass of the pendulum.

The point in the line pC, Fig. 61, at a distance l from p is called the *center of oscillation* of the pendulum. This point is also called the *center of percussion* of the pendulum, for the reason that if the pendulum is started or stopped by a horizontal hammer blow at this point no side force is produced on the supporting axis. See article 109.

106. The determination of gravity. — The most accurate determination of the acceleration of gravity is made by means of the pendulum. This determination would be a very simple thing if it were feasible to construct a simple pendulum, in which case equation (59)*a* could be used for calculating gravity from the measured length l of the simple pendulum and its observed period τ.

The determination of the acceleration of gravity by means of an actual pendulum depends in general upon the determination of the moment of inertia of the pendulum as is evident from equation (58), and the moment of inertia of a body cannot be determined with great accuracy. This difficulty is obviated by means of the so-called *reversion pendulum* which is due to Kater (1818).

Theory of the reversion pendulum. — Consider a body of mass m, its center of mass at O, Fig. 62. Let O', O, and O'' be co-linear points;

let τ' and τ'' be the vibration periods of the body swung as a pendulum from O' and O'' respectively; and let K, K', and K'' be the moments of inertia of the body about O, O', and O'' respectively. From equation (58) we have

$$\frac{4\pi^2 K'}{\tau'^2} = mgx \qquad \text{(i)}$$

and

$$\frac{4\pi^2 K''}{\tau''^2} = mgy \qquad \text{(ii)}$$

From equation (52) we have

$$K' = K + x^2 m \qquad \text{(iii)}$$

$$K'' = K + y^2 m \qquad \text{(iv)}$$

Substituting these values of K' and K'' in (i) and (ii), we have

$$\frac{4\pi^2(K + x^2 m)}{\tau'^2} = mgx \qquad \text{(v)}$$

$$\frac{4\pi^2(K + y^2 m)}{\tau''^2} = mgy \qquad \text{(vi)}$$

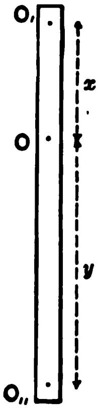

Fig. 62.

Eliminating K/m from (v) and (vi), we have

$$\frac{4\pi^2(x^2 - y^2)}{x\tau'^2 - y\tau''^2} = g \qquad (61)$$

If $\tau' = \tau''$, we may cancel $(x - y)$, *if* $(x - y)$ *is not equal to zero,* giving

$$\frac{4\pi^2(x + y)}{\tau'^2} = g \qquad (62)$$

(1) If the pendulum has been adjusted by repeated trial, so that $\tau' = \tau''$, then equation (62) enables the calculation of g when $(x + y)$ and τ' have been observed.

(2) If the pendulum has not been adjusted, equation (61) enables the calculation of g when x, y, τ', and τ'' have been observed.

(3) If the pendulum has been roughly adjusted, so that τ' and τ'' are nearly equal, then *equal and opposite errors in* x *and* y *very nearly annul each other in their influence upon the value of* g *as calculated by equation* 61. Therefore equation (61) gives g very accurately when τ' and τ'' are nearly equal, $(x + y)$ being measured with great accuracy, and x measured roughly. The value of y is taken from $(x + y) - x$, so that its error may counteract the error due to the roughly measured value

of x. The position of the center of mass O, Fig. 62, is found with sufficient accuracy for this rough measurement of x, by balancing the pendulum horizontally on a knife edge.

Note.—When $x = y$, equation (61) becomes indeterminate, giving $0/0 = g$, and in this case equation (62) is not necessarily true, since it has been derived from equation (61) by cancelling $(x - y)$, which is zero.

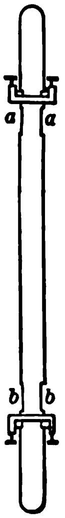

Fig. 63.

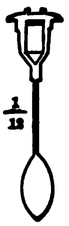

Fig. 64.

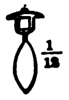

Fig. 65.

In its most approved form Kater's reversion pendulum is so designed as to be entirely symmetrical with respect to the two knife edges aa and bb as shown in Fig. 63, and one end of the pendulum is loaded with lead placed in the interior, so that the center of mass may not be midway between the knife edges. The advantage of symmetry of form is that the influence of the surrounding air is then nearly the same whether the pendulum is suspended from aa or from bb, and it is easier to make approximate allowance for errors due to the action of the air.

107. Comparison of the values of gravity at two places by means of the pendulum.—If one and the same pendulum is swung at two places in succession and its respective periods τ and τ' observed, we have from equation (58)

$$\frac{4\pi^2 K}{\tau^2} = mgx \tag{i}$$

and

$$\frac{4\pi^2 K}{\tau'^2} = mg'x \tag{ii}$$

in which g and g' are the respective values of the acceleration of gravity at the two places. Dividing equation (i) by equation (ii) member by member, we have

$$\frac{g}{g'} = \frac{\tau'^2}{\tau^2} \tag{63}$$

From this equation the value of g may be accurately determined at any place in terms of its known value at another place by observing the values of τ and τ' of an ordinary pendulum, every precaution being taken to avoid variations of dimensions of the pendulum due to temperature or to careless handling. Most of the gravity determinations of the U. S. Coast and Geodetic Survey are made in this way, the value of g at Washington having been once for all determined with the greatest possible accuracy by means of Kater's pendulum. The portable pendulums * used for this purpose are shown in Figs. 64 and 65.

The following are some of the results of the experimental determination of g:

TABLE.

Locality.	Latitude.	Longitude.	Elevation.	Value of g (not Reduced to Sea-level).
Boston, Mass.	42° 21′ 33″	71° 03′ 50″	22 meters.	980.382
Philadelphia, Pa.	39 57 06	75 11 40	16 "	980.182
Washington, D. C.	38 53 20	77 01 32	10 "	980.100
Ithaca, N. Y.	42 27 04	76 29 00	247 "	980.286
Cleveland, O.	41 30 22	81 36 38	210 "	980.227
Cincinnati, O.	39 08 20	84 25 20	245 "	979.990
Terre Haute, Ind.	39 28 42	87 23 49	151 "	980.058
Chicago, Ill.	41 47 25	87 36 03	182 "	980.264
St. Louis, Mo.	38 38 03	90 12 13	154 "	979.987
Kansas City, Mo	39 05 50	94 35 21	278 "	979.976
Denver, Col.	39 40 36	104 56 55	1638 "	979.595
San Francisco, Cal.	37 47 00	122 26 00	114 "	979.951
Greenwich.	51 29 00	0 00 00	47 "	981.170
Paris.	48 50 11	2 20 15	72 "	980.960
Berlin	52 30 16	13 23 44	35 "	981.240
Vienna.	48 12 35	16 22 55	150 "	980.852
Rome.	41 53 53	12 28 45	53 "	980.312
Hammerfest	70 40 00	22 38 00	—	982.580

108. Conjugate relation between point of support and center of oscillation of a pendulum. — Solving equation (62) for $(x+y)$ and putting τ for the common value of τ' and τ'', we have $x+y=\tau^2 g/4\pi^2$, which, being identical to $l=\tau^2 g/4\pi^2$ (59)*b*, shows that O'' is the center of oscillation of the pendulum when O' is the support and *vice versa*.

109. Identity of center of oscillation and center of percussion of a pendulum. — Consider a pendulum suspended at O', Fig. 66, with its center of mass at O distant x below O', and its center of percussion at O'' distant l below O'. If the pendulum turns about O' at angular velocity ω, then $v = r\omega$ is the linear velocity of a particle Δm of the pendulum distant r below O', and $\dot{v} = r\dot{\omega}$ is the acceleration of this particle, so that $F = \Delta m \cdot \dot{v} = r\dot{\omega} \cdot \Delta m$ is the side force necessarily acting upon the particle Δm. Similar forces must act on all the particles of the pendulum. The resultant of all such forces must pass through the center of percussion, for that is, by definition, the point at which a *single* force acting upon the pendulum will start it turning about O'; therefore the total torque action of these forces about O'' must be zero. That is, $\Sigma(l - r)r\dot{\omega} \cdot \Delta m = 0$, or $l\Sigma r \cdot \Delta m = \Sigma r^2 \Delta m$. But $\Sigma r^2 \Delta m = K$ is the moment of inertia of the pendulum about O' and $\Sigma r \cdot \Delta m = mx$ from equation (21),* m being the total mass of the pendulum. Therefore $lmx = K$, or $l = K/mx$, which being identical to equation (60) shows that O'', the center of percussion of the pendulum, is also the center of oscillation of the pendulum.

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* Mendenhall, Report of the U. S. Coast and Geodetic Survey, 1891, II., p. 504; also G. R. Putnam, Bulletin Philosophical Society of Washington, Vol. XIII., p. 31.

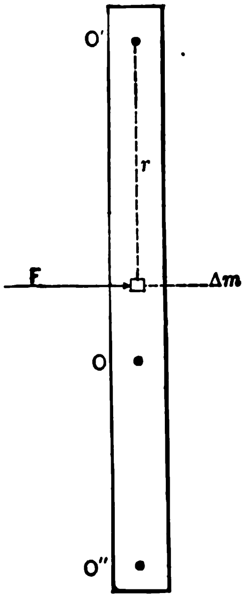

Fig. 66.

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* The point O' being the origin of coördinates, it is evident that $O'O$ $(= x)$ is what is represented by X in equation (21), while what is here represented by r is represented by x in equation (21).

CHAPTER VI.

THEORY OF ELASTICITY. FRICTION.

110. Stress and strain. — When external forces act upon a body and tend to change its shape, internal forces are brought into action between contiguous parts of the body throughout. This force action between contiguous parts of a body is called *stress*.

A body under stress is always distorted; such distortion is called *strain*.

111. Definition of stretch. — The simplest type of strain is that in which the linear dimensions of a body are made to increase (or decrease) parallel to a given direction, without any change of dimensions in directions at right angles to the given direction. Such a strain is called a *stretch*. The given direction is called the *axis* of the stretch.

Let l be the initial length of a body, and Δl its increment of length when stretched; the ratio $\Delta l/l$ is adopted as the measure of the stretch. Throughout this chapter the ratio stretched length, $l + \Delta l$, divided by initial length l is expressed as $1 + f$. That is:

$$1 + f = \frac{l + \Delta l}{l}$$

or

$$1 + f = 1 + \Delta l/l$$

whence

$$f = \Delta l/l \tag{64}$$

Positive and negative stretches. — When the length of a body is increased Δl is positive and the value of $\Delta l/l$ is positive. When the length ot a body is decreased Δl is negative and the value of $\Delta l/l$ is negative. Increase of length is called a positive stretch; decrease of length is called a negative stretch.

Example. — The initial length of a body is 1000 units and it is stretched to a length of 1015 units. The stretch is then 0.015, that is is, fifteen parts in a thousand of initial length. In the theory of elasticity all stretches are supposed to be small.

112. Principal stretches of a strain.*—A small spherical portion of a body becomes an ellipsoid when the body is strained.†

A distortion which changes a sphere into an ellipsoid is always equivalent to three mutually perpendicular stretches, parallel, if there has been no rotation, to the axes of the resulting ellipsoid.‡

The three stretches of a strain are represented by the letters f, g, and h.

113. The strain ellipsoid.—The semi-axes of the actual ellipsoid into which a small spherical portion, of radius r, of a body is distorted by a strain are $(1+f)r$, $(1+g)r$, and $(1+h)r$ respectively. An ellipsoid similarly related to a sphere of *unit* radius, having semi-axes equal to $(1+f)$, $(1+g)$, and $(1+h)$, is called the *strain ellipsoid*. This strain ellipsoid in its relation to the unit sphere is a complete geometrical representation of the distortion of the small spherical portion of the body.

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* Sometimes called principal dilatations.

† *Proof.* — Let the origin of coördinates be chosen at the center of the small spherical portion of the body. Let $x^2+y^2+z^2=r^2$ be the equation to the surface of this small sphere. Let x', y', and z' be the coördinates of the point xyz after strain. If there is no rupture in the neighborhood, x', y', and z' are continuous functions of x, y, and z, and may each be expanded in a series of ascending powers of x, y, and z. Since x, y, and z are very small, squares and higher powers may be neglected, giving

$$x' = a_1x + b_1y + c_1z$$
$$y' = a_2x + b_2y + c_2z$$
$$z' = a_3x + b_3y + c_3z$$

in which the a's, b's, and c's are constants. These equations transform the equation to the small sphere into an equation still of the second degree, which can be shown to be the equation to an ellipsoid.

‡ This proposition is a generalization of the one to the effect that the distortion which changes a circle into an ellipse is always equivalent to two mutually perpendicular stretches.

Axes of strain.—The directions of the three stretches of a strain, or the directions of the axes of the strain ellipsoid are called the *axes* of the strain.

114. Homogeneous strains. — A body is said to be homogeneously strained when every part of it is distorted in precisely the same way. The strain ellipsoids in all parts of a body homogeneously strained are similar in shape and in orientation.

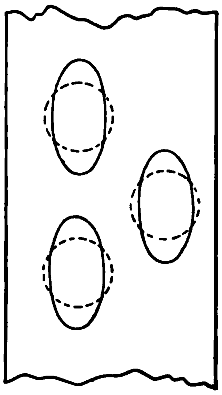

Fig. 67.

A stretched wire or rod is homogeneously strained. The small ellipses in Fig. 67 show the way in which spherical portions (represented by the dotted circles) of a stretched rod are distorted.

A bent beam is non-homogeneously strained. Fig. 68 shows a bent beam. Along the line *mm* there is no distortion. The portions of the beam above *mm* are distorted as shown by the small circles and ellipses *aa*. The portions of the beam below *mm* are distorted as shown by the small circles and ellipses *bb*.

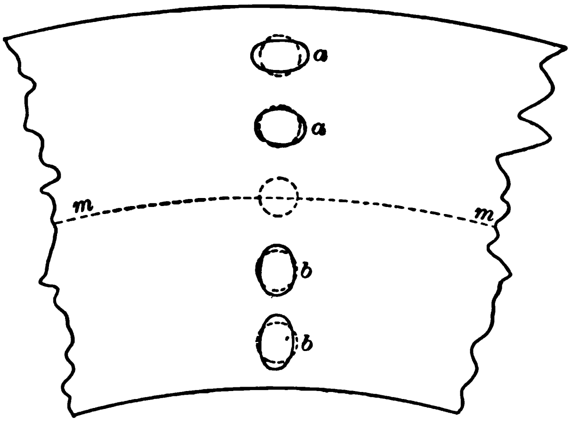

Fig. 68.

Remark. — All that is hereafter said of stresses and strains applies without restriction to bodies homogeneously strained; and to the *small parts* of bodies non-homogeneously strained. See Art. 29.

115. Distortion of a cubical portion of a body. — A cubical portion of a body is changed by a strain to a parallelepiped. If the edges of the cube are parallel to the axis of the strain the parallelepiped will be rectangular, otherwise not.

116. Specification of a strain. — A strain which is unaccompanied by rotation is completely specified by its three stretches f, g, and h, together with the three angles necessary to specify the directions of these stretches.

117. Example of strain in two dimensions. — A thin sheet of rubber is stretched over a frame and held in place by spring clips. This state of the rubber sheet is taken as the normal or unstrained state, and small squares and circles (also a large circle) are marked upon it. The rubber is now stretched in any manner whatever, and all the small circles are changed to ellipses, while all the small squares are changed to parallelograms. If a parallelogram has its sides parallel to the axes of a nearby ellipse, then the parallelogram is a rectangle, otherwise not. If the small ellipses are not similar or if their axes are not all in the same two directions, the rubber sheet is non-homogeneously strained, that is dissimilarly strained in its various parts. In this case the large circle will be distorted out of all semblance to an ellipse, while the small circles *even when greatly magnified* by a microscope appear to be sensibly elliptical.

STRESS.

118. Stress on a section. — Imagine a small plane area q, called a *section*, in the interior of a body under stress. The portions of the body on the two sides of this section pull on each other, or push on each other, or tend to slide over each other. Let a be the force with which the portions of the body on the two sides of the section act on each other across the section. The force per unit area, a/q, is called the *stress on the section.*

When the force a is normal to the section the stress is called a *pull*. A pull is considered positive when the material on one side of the section actually pulls on the material on the other side. When the material on one side of a section pushes on the material on the other side we have what is considered a negative pull. When the line of action of the force a lies in the plane of the section the stress on the section is called a *tangential stress on the section*.

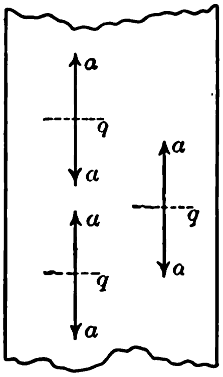

Fig. 69.

119. Homogeneous stress. — A body is said to be under homogeneous stress when the stress is precisely the same in all parts of the body. In this case the stress on all parallel sections is the same in magnitude and direction.

The stress in a stretched rod is homogeneous. The dotted lines q, q, q, q, Fig. 69, represent parallel sections in a stretched rod and the arrows a, a, a, etc., represent equal pulls.

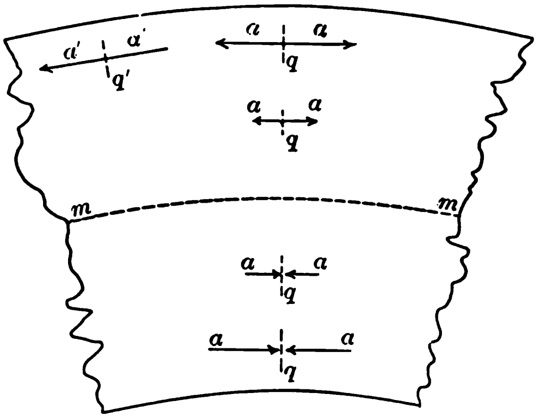

Fig. 70.

The stress in a bent beam is non-homogeneous. The dotted lines q, q, q, q, Fig. 70, represent parallel sections in a bent beam and the arrows a, a, a, etc., represent the unequal pulls (or pushes) on the sections.

120. Principal Sections of a Stress. Pulls of a Stress. The conditions of equilibrium of a small portion of a body, as expressed by equations (36) and (39) lead * to the result that at a point in a body under stress there must be three mutually perpendicular sections upon which the stress is normal. That is, *the stress in the neighborhood of a point in a body consists of three mutually perpendicular pulls.* These sections are called the *principal sections* of the stress. The three pulls of a stress are represented by the letters F, G, and H.

Axes of a stress. — The directions of the three pulls of a stress are called the axes of the stress.

121. Specification of Stress. — A stress is completely specified by its three **pulls,** together with the three angles necessary to specify their directions.

122. Problem. — *To find the stress on any section when the stress is specified as above, and the direction of the normal to the given section is specified.* Let OX, OY, and OZ (Fig. 71) be the axes of the stress; F, G, and H the pulls parallel to X, Y, and Z respectively; XYZ the given section of area q. Let l, m, and n be the angles between the normal to the given section, and the axes X, Y, and Z respectively. Consider the material inside the tetrahedron $XYZO$. It is in equilibrium; therefore [equations (36)] the forces acting on this material across the faces YOZ, XOZ, and XOY are equal and opposite to the respective components of the force acting across the given section XYZ. Dividing these components of the force acting across XYZ by q, we have the component of the required stress on XYZ. The areas of YOZ, XOZ, and XOY are $q \cos l$, $q \cos m$, and $q \cos n$ respectively. The forces acting across these faces are $-Fq \cos l$, $-Gq \cos m$, and $-Hq \cos n$, respectively; F, G, and H are *pulls*, and these forces are considered negative, since they are in the negative directions of the respective axes. The components of the stress on the section XYZ are, therefore, $F \cos l$, $G \cos m$, and $H \cos n$.

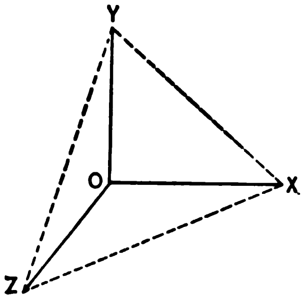

Fig. 71.

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* For details of proof see article *Elasticity, Theory of*, Chapter III., in Ency. Brit., 9th edition.

RELATIONS OF STRESS AND STRAIN.

123. Corresponding Stress and Strain. — A definite strain is always associated in a given body with a given stress. A stress and a strain which are associated in this way are said to correspond to each other. The theoretical and experimental study of elasticity has to do mainly with the detailed character of this correspondence of stress and strain.

124. Isotropy and aelotropy. — A body is said to be isotropic when it has identically the same physical properties in all directions; otherwise it is said to be *aelotropic*. Annealed glass, cast metal, liquids and gases are isotropic; rolled metal, wood, and many crystals are aelotropic. The discussion of the elasticity of aelotropic bodies is beyond the scope of this text.

The axes of stress and strain coincide in isotropic bodies. This is exemplified in two dimensions by the stress and strain in a sheet of rubber. When such a sheet is stretched in any way whatever a small circular portion of the sheet is changed to an ellipse, and the force action between contiguous portions of the sheet consists of two simple pulls across the axes of the ellipse.

125. Hooke's law. — Let F, G and H be the three pulls of a given stress and f, g and h the three stretches of the corresponding strain. If the stress is multiplied n times so that the pulls become nF, nG, and nH; then the strain is multiplied n times and the stretches become nf, ng, and nh. Stress and strain are proportional.

Hooke's law is quite accurately true for some substances when stress and strain are not large. The limitations to Hooke's law are pointed out in article 148.

Examples. — The flexure of a beam is proportional to the load on the beam. The elongation of a rod or wire is proportional to the stretching force. The twist of a shaft is proportional to the twisting force or torque. The elongation of a helical spring is proportional to the stretching force.

126. Types of Stress and Strain. — The relative values of the three pulls of a given stress (or the relative values of the three stretches of a given strain) determine to what *type of stress* (or strain) the given stress (or strain) belongs. The important types of stress and strain are longitudinal stress, hydrostatic pressure (a stress), and shearing stress, and the corresponding strains.

127. Longitudinal Stress. — A longitudinal stress is a stress having one pull finite and the other two equal to zero. The value P of the finite pull is adopted as the numerical measure of the stress. The direction of the finite pull is the only determinate axis and it is called *the axis* of the longitudinal stress.

128. Longitudinal Strain. — A longitudinal strain is the strain which in isotropic bodies accompanies longitudinal stress. *All three stretches of a longitudinal strain are finite.* The stretch β, in the direction of the axis is adopted as the measure of the strain. The other two stretches are equal. They are always opposite in sign to the axial stretch and they bear to that stretch a constant ratio, for a given substance. This ratio is called *Poisson's ratio.*

129. Stretch Modulus * of a Substance. — According to Hooke's law the axial stretch produced by a longitudinal stress is proportional to the stress so that the ratio: longitudinal stress divided by the longitudinal strain is, for a given substance, a constant. Therefore we may write

$$E = \frac{P}{\beta} \tag{65}$$

in which β is the stretch produced by a pull P in its own direction. The constant E is called the stretch modulus of the substance.

130. General equations of stress and strain.—Let F be a longitudinal stress, that is a simple pull, in the direction of the x-axis of reference, and let f', g', and h' be the three stretches, parallel to x, y, and z axes respectively, produced by F. Then we have:

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* Often called Young's Modulus.

$$\begin{aligned} f' &= aF \\ g' &= -bF \\ h' &= -bF \end{aligned} \qquad \text{(i)}$$

in which a and b are constants for a given substance. The basis, in fact, of equations (i) may be seen by reading articles 125, 127, and 128. The constant a in equations (i) is equal to the reciprocal of the stretch modules E, and b/a is Poisson's ratio.

Similarly, let G be a longitudinal stress in the direction of the y-axis of reference, and let f'', g'', and h'' be the three stretches, parallel to x, y, and z axes respectively, produced by G. Then we have:

$$\begin{aligned} f'' &= -bG \\ g'' &= aG \\ h'' &= -bG \end{aligned} \qquad \text{(ii)}$$

Similarly, let H be a longitudinal stress parallel to the z-axis of reference, and let f''', g''', and h''' be the three stretches, parellel to the x, y, and z axes, respectively, produced by H. Then we have:

$$\begin{aligned} f''' &= -bH \\ g''' &= -bH \\ h''' &= aH \end{aligned} \qquad \text{(iii)}$$

Experiment shows * that the stretch produced in any direction by a number of pulls acting together is equal to the sum of the stretches in that direction produced by the respective pulls acting separately. Therefore:

$$\begin{aligned} f &= f' + f'' + f''' \\ g &= g' + g'' + g''' \\ h &= h' + h'' + h''' \end{aligned} \qquad \text{(iv)}$$

---

* In general, any effect which is *proportional* to a cause, may be resolved into parts which correspond to the parts of the cause. Thus a spring stretches in proportion to the stretching force. One kilogram produces, say, one centimeter elongation; two kilograms produce two centimeters elongation, which is one centimeter for each kilogram.

in which f, g, and h are the stretches, parallel to the x, y, and z axes, respectively, produced by the three pulls F, G, and H acting together. Therefore substituting the values of f', f'', f''', g', g'', g''', h', h'', and h''' from equations (i), (ii), and (iii) in equation (iv), and we have:

$$\begin{aligned} f &= aF - bG - bH \\ g &= -bF + aG - bH \\ h &= -bF - bG + aH \end{aligned} \tag{66}$$

These equations give the strain, of which the three stretches are f, g, and h, produced in an isotropic substance by any given stress of which the three pulls are F, G, and H; a and b being constants for a given substance.

131. Hydrostatic pressure. — Hydrostatic pressure is a stress in which all three pulls are equal. The common value, p, of these pulls is adopted as the measure of the pressure.

132. Pascal's principle. — In a substance under hydrostatic pressure the stress on **all** sections is normal and of the same value.

Proof. — In the problem of Art. 122 put $F = G = H = p$. Then the components of the stress on the section XYZ become $p \cos l$, $p \cos m$ and $p \cos n$, which are evidently the components of a stress which is equal to p and which makes angles l, m, and n with the axes X, Y, and Z respectively, as does the normal to the section.

Corollary. — A stress such that on no section is there any tangential stress is an hydrostatic pressure.

133. Isotropic Strain. — Isotropic strain is a strain having its three stretches f, g, and h equal. It is the strain which in an isotropic substance corresponds to hydrostatic pressure.

Measure of isotropic strain. — The value of an isotropic strain could be specified by the common value, i, of the three stretches f, g, and h, but it is customary to use the ratio $\Delta v/v$ as the measure of isotropic strain, Δv, being the change in volume of a portion of the body of which the initial volume is v.

Proposition. — The change of volume per unit volume $(\Delta v/v)$ of a body under isotropic strain is equal to three times its change of length per unit length $(\Delta l/l = i)$. That is:

$$\frac{\Delta v}{v} = 3i \qquad (67)$$

Proof. — Consider a cubical portion of the body, the initial length of edges of cube being l. After the strain the length of edge becomes $l(1 + i)$. Therefore $v = l^3$ and $v + \Delta v = l^3(1 + i)^3 = (1 + 3i + \cdots)$. Discarding terms in i^2 and i^3 which are negligible, since i is very small, we have

$$v = l^3$$

$$v + \Delta v = l^3(1 + 3i)$$

whence equation (67) follows directly. Q. E. D.

134. Bulk Modulus* of a Substance. — According to Hooke's law the ratio hydrostatic pressure divided by the corresponding strain is a constant for a given substance. This constant is called the ***bulk modulus*** of the substance. Using $\Delta v/v$ as the measure of isotropic strain we have therefore:

$$V = \frac{P}{\Delta v/v} \qquad (68)$$

in which V is the bulk modulus of a substance, v is its initial volume, and Δv is the change of volume produced by an hydrostatic pressure p.

135. Relations between stretch modulus, Poisson's ratio, and bulk modulus. — Putting $F = G = H = p$ in equations (66), and also $f = g = h = i$, these equations give

$$i = (a - 2b)p$$

whence

$$3i = \Delta v/v = 3(a - 2b)p$$

whence

$$\frac{p}{\Delta v/v} = V = \frac{1}{3a - 6b} \qquad (69)$$

in which $1/a$ is the stretch modulus and b/a is Poisson's ratio for the given substance.

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* The reciprocal of bulk modulus is sometimes called the coefficient of compressibility of a substance.

136. Shearing stress. — A shearing stress is a stress having two of its pulls equal and opposite, and the third equal to zero The common numerical value of the two pulls is adopted as the measure of a shearing stress.

Consider a cubical portion $ABCD$ of a body under shearing stress, the faces of the cube being principal sections of the stress. The stress on the faces AB and CD, Fig. 72, is a *push* S*;* on the faces AC and BD an equal *pull.* On the faces of the cube, parallel to the paper, the stress is zero. The plane $ABCD$ (here the plane of the paper), is called *the plane of the shearing stress.*

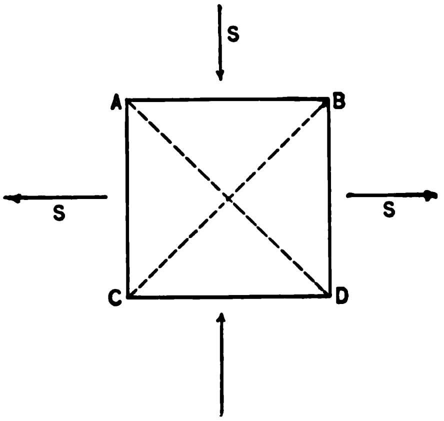

Fig. 72.

Proposition. — *The stress on the diagonal sections AD and CB, Fig. 72, is purely tangential and equal to S.*

Proof. — Let l be the length of an edge of the cube. Consider the portion ABD of the cube. It is in equilibrium, and the resultant of the forces acting across the faces AB and BD, ignoring sign, is equal to, and has the same line of action as, the force acting across the face AD. The area of each AB and BD is l^2, and of AD is $l^2\sqrt{2}$. The resultant of the forces Sl^2 and Sl^2, which act across AB and BD, is a force *lying in the line* AD, and equal to $Sl^2\sqrt{2}$, which, divided by the area of AD, gives S. Q.E.D.

137. Shearing strain. — A shearing strain is a strain having two of its stretches equal and opposite, and the third equal to zero. It is the strain which in isotropic bodies accompanies shearing stress.

Measure of shearing strain; angle of shear. — Let the heavy line square, Fig. 73, represent a cubical portion of a body which is changed under the shearing strain into a rectangular parallelepiped represented by the dotted rectangle. Dimensions perpendicular to the plane of the paper remain unaltered. The dotted rhombus is a figure which was square in the unstrained cube.

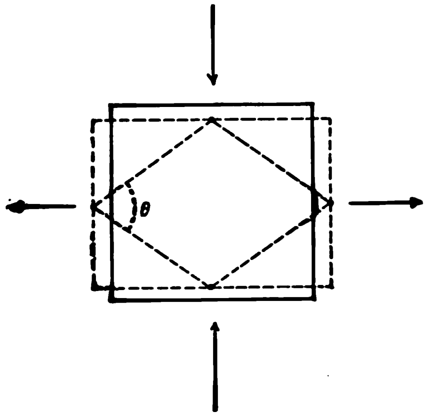

Fig. 73.

The angle $\phi = \pi/2 - \theta$ is called the *angle of the shearing strain* and is adopted as the measure of the strain.

Proposition. — *The angle ϕ of a shearing strain is equal to twice the stretch, α, of the strain.*

Proof. — Let the rhombus $eCBD$, Fig. 74, be the dotted rhombus of Fig. 73 *turned* so that its side DB coincides with the side of the square from which it was distorted by the strain. Let l be the length of AD. Then *the length of the diagonal AB is $l\sqrt{2}$.* Since ϕ is small, $Ae = l\phi$, and since Aef is sensibly a right triangle with equal legs, Af is equal to $Ae/(\sqrt{2})$. Further, the shortened diagonal eB of the rhombus is sensibly equal to $AB - Af$, therefore *the length of the shortened diagonal* *eB of the rhombus is* $l\sqrt{2} - l\phi/\sqrt{2}$. From the definition of stretch we have $1 - \alpha = eB/AB$. Substituting in this the above values of AB and eB, we have

$$\alpha = \tfrac{1}{2}\phi \tag{70}$$

138. Slide modulus of a substance.—According to Hooke's law the ratio: Shearing stress divided by shearing strain is a constant for a given substance. This constant is called the *slide modulus* of the substance. Therefore we have:

$$n = \frac{S}{\phi} \tag{71}$$

in which n is the slide modulus of a substance and ϕ is the angle of the shearing stress S.

139. Relation between stretch modulus, Poisson's ratio, and slide modulus. — Putting, in equations (66), $F = S$, $G = -S$, $H = 0$, $f = \alpha$, $g = -\alpha$, and $h = 0$, we have:

$$\alpha = (a + b)S$$

whence

$$2\alpha = \phi = 2(a + b)S$$

whence

$$\frac{S}{\phi} = n = \frac{1}{2(a + b)} \tag{72}$$

in which $1/a$ is the stretch modulus, and b/a is Poisson's ratio for the given substance.

140. Potential energy of strain.—The potential energy per unit volume of a strained body is equal to one half the product of the stress into the strain, or to one half the product of the modulus into the square of the strain.

Proof for longitudinal strain. — Consider a cube, edge l, of the substance under longitudinal stress P, stretched to length $l + m$, in the direction of P, m being small. From the definition of stretch we have $\beta = m/l$ or $d\beta = dm/l$, β being the measure of the longitudinal strain as per Art. 128. Let P increase slightly, causing a further increase dm in the length of the cube.

The work dW, done in producing this increment dm of length is Pl^2dm, for the area across which the pull P acts is sensibly l^2, and the stretching force Pl^2 acts through the distance dm. Therefore $dW = Pl^2dm = Pl^3dm/l = Pl^3d\beta$. In this expression write v (volume of cube) for l^3 and substitute the value of P from $E = P/\beta$ (65), and we have $dW = vE\beta d\beta$. Since v is sensibly constant and $W = 0$ when $\beta = 0$, we have by integration

$$W = \tfrac{1}{2}vE\beta^2$$

Substituting P/β for E from (65), we have

$$W = \tfrac{1}{2}vP\beta$$

EXAMPLES OF STRESS AND STRAIN.

141. Hydrostatic pressure in a heavy liquid. — A heavy liquid under the action of gravity furnishes an example of a non-homogeneously distributed hydrostatic pressure. To find the hydrostatic pressure at a point we need only find the force acting on any section at the point since by Pascal's principle the stress is the same on all sections. Let us choose a horizontal section SS', Fig. 75, of area a at a distance x beneath the surface of a liquid of density d. The force, F, on the section SS' is due to the weight of liquid above it. The volume of this liquid is ax, its mass is axd, and its weight is $axdg$ dynes. Therefore the force, F, on the section SS' is equal to $axdg$ dynes, so that the force per unit area is xdg. Therefore

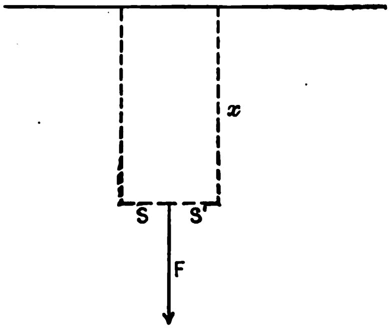

Fig. 75.

$$p = xdg \qquad (73)a$$

If the pressure at the surface of the liquid is p_0 then

$$p = p_0 + xdg \qquad (73)^b$$

in which p is the hydrostatic pressure at a point distant x beneath the surface of a liquid of density d.

142. Stretched wire. — A wire of sectional area q, held tense by a force F, is under homogeneous longitudinal stress P such that

$$P = \frac{F}{q} \qquad (74)$$

Let l be the initial length of the wire, and $l + \Delta l$ its length under tension, then

$$l : l + \Delta l = 1 : 1 + \beta;$$

whence $\beta = \Delta l / l$. Substituting this value of β, and the above value of P in $E = P/\beta$ (65), we have

$$E = \frac{Fl}{q\Delta l} \qquad (75)$$

This equation enables the calculation of E when F, l, q, and Δl have been observed.

The wire to be experimented upon is fastened at one end to a wall bracket, Fig. 76, or other firm support, and a weight is applied at the free end. The wire thus rendered vertical and tense is further stretched by a known force F.

Reading microscopes are focused upon points a and b near the upper and the lower ends of the wire. The difference in the movements of the two points is the elongation Δl. The quantity l, which is the distance ab, is found by direct measurement;

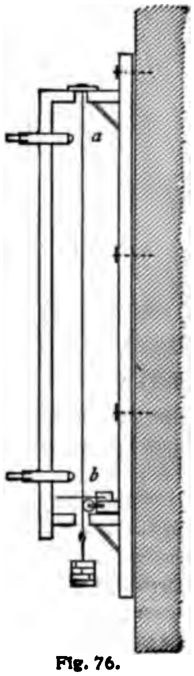

Fig. 76.

while q is computed from the diameter, or from the density of the material and the weight (per meter) of the wire.

If F be expressed in dynes, l and Δl in centimeters and q in square centimeters, equation (75) gives the modulus in c. g. s. units. The following table gives the value of the stretch modulus for several materials in c. g. s. units.

TABLE.

STRETCH MODULUS OF VARIOUS MATERIALS.

DYNES PER SQUARE CENTIMETER.

Brass	0.83×10^{12}	to 0.97×10^{12}
Copper	0.98	to 1.0
Gold	0.52	to 0.79
Iron	1.7	to 2.4
Lead	0.15	to 0.18
Platinum	1.5	to 0.7
Silver	0.7	to 0.71
Zinc	0.85	
Glass	0.66	to 0.78
Birch	0.091	
Oak	0.092	
Poplar	0.05	
Spruce	0.11	

143. Bent beam.* — Consider such a portion of a slightly bent beam, Fig. 77, as subtends a small angle θ from its center of curvature C. All the filaments of the beam in the short portion are assumed to be circular and concentric. The filaments in the upper part of the beam are stretched, those in the lower part are shortened. An intermediate filament retains its initial length. This filament marks the so-called median line of the beam represented in the figure by the dotted line.

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* This discussion of a bent beam is incorrect, although it is the one usually given in elementary treatises on Mechanics. It is accurate enough for most purposes. The incorrectness is evident when we consider that the assumed strain, having no lateral stretch, is not a true longitudinal strain. There is nothing *in itself* which shows the assumed distortion of a beam to be impossible, but if the *pulls* at each point necessary to produce the assumed simple elongation be calculated, it will be found that the *vertical* pulls cannot balance each other in the different parts of the beam, and the particles of the beam will not therefore be in equilibrium. To calculate the *pulls* necessary to produce simple elongation, take f, equations (66), as given, and put $g = h = 0$.

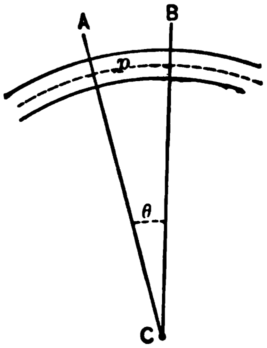

Fig. 77.

Let R be the radius of curvature of the median line; then $R\theta$ is the initial length of all filaments between AC and BC. The length of the filament at a distance x above the median line is $(R+x)\theta$.

Therefore $1:1+\beta = R\theta:(R+x)\theta$,

whence
$$\beta = \frac{x}{R} \tag{76a}$$

in which β is the stretch of the filament.

Substituting this value of β in (65) and solving for P, we have

$$P = \frac{Ex}{R} \tag{76b}$$

In this equation P is the longitudinal stress in that part of the beam distant x above the median line.

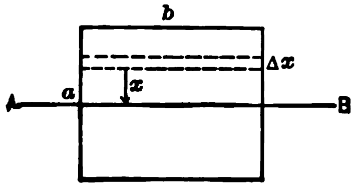

Fig. 78.

144. Proposition. — The total force action across a complete section of a bent beam is a torque T about a line p, Fig. 77, perpendicular to the plane of the paper, such that

$$T = \frac{1}{12}\frac{a^3bE}{R} \tag{77}$$

in which a is the depth of the beam and b is its width.

Proof. — Let Fig. 78 represent a cross-section of the beam. The force ΔF acting across the area $b\cdot\Delta x$, of the narrow dotted strip, is $\Delta F = Pb\cdot\Delta x$, $\therefore$ using value of P from (76)b,

$$\Delta F = \frac{Eb}{R} x \cdot \Delta x$$

The torque ΔT of this force about the line AB is

$$\Delta T = \frac{Eb}{R} x^2 \cdot \Delta x$$

whence the torque across the complete section is

$$T = \frac{Eb}{R} \int_{-\frac{a}{2}}^{+\frac{a}{2}} x^2 dx = \frac{1}{12} \frac{a^3 bE}{R}$$

145. Twisted Rod. — Consider those portions of a twisted rod which lie in the *surface of a cylinder* coaxial with the wire and of radius r — less than the radius of the wire. Let l be the length of the wire, and θ the angle through which one end of it is turned. Let Figs. 79 and 80 represent the development of this cylindrical surface, Fig. 79 before twisting and Fig. 80 after twisting. The line AB is brought into the position $A'B'$, and the small square is distorted into the rhombus by the twist. The portions of the wire under consideration are thus subjected to a *shearing strain*, of which the angle ϕ, as is evident from the figure, is

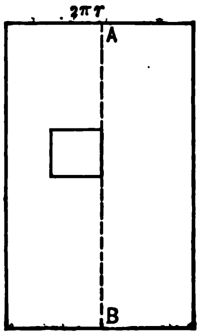

Fig. 79.

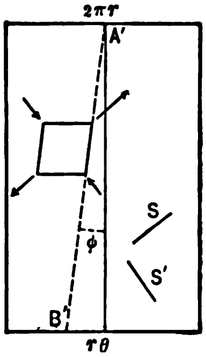

Fig. 80.

$$\phi = \frac{r\theta}{l} \qquad (78)a$$

Substituting this value of ϕ in $n = \frac{S}{\phi}$(71), and solving for S, we have

$$S = \frac{nr\theta}{l} \tag{78b}$$

in which S is the shearing stress in the portion of the wire under consideration. Comparing Fig. 80 with figures in articles 136 and 137, we see that the stress at a point in a twisted wire is such that (1) on all sections perpendicular to the axis of the wire the stress is purely tangential, and perpendicular to the radius; (2) on all sections of which the planes include the axis of the wire the stress is purely tangential and parallel to the axis of the wire; (3) on all sections which are perpendicular to a radius the stress is zero; (4) on such sections as S, Fig. 80, the stress is a push, and (5) on such sections as S', Fig. 80, the stress is a pull. In every case the value of the stress is that given by equation (78)*b*.

This discussion of the strain in a twisted rod may be made more easily intelligible by means of a model as follows: A tin cylinder about 20 cm. in diameter and 35 cm. long has wooden disks fixed in each end. On one end is an additional wooden disk which turns on a nail which is in the axis of the cylinder. Loosely woven muslin is tacked at one end to the moveable disk and to the fixed disk at the other end. This muslin fits the tin cylinder closely and the seam at one side is sewed. On this muslin a small square may be drawn like Fig. 79, and also small circles, and when the movable disk is turned through a considerable angle, the distortion of the square and of the circles will give a clear idea of the character of the strain in a cylindrical shell of a twisted rod.

146. Constant of torsion of a rod or wire, determination of slide modulus. — *The total force action across a complete section of a twisted rod is a torque T about the axis of the rod, and the value of this torque is:*

$$T = -\frac{\pi n \rho^4 \theta}{2l} \tag{79}$$

in which n is the slide modulus of the material of the rod, ρ is the radius of the rod, l is the length of the rod, and θ is the angle through which one end of the rod is twisted. The negative sign is written for the reason that the torque tends to reduce θ, that is T and θ are opposite in sign.

This equation (79) shows that T is proportional to θ. The proportionality factor $\pi n \rho^4/2l$ is called the constant of torsion of the rod or wire.

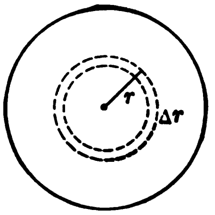

Fig. 81.

This equation (79) enables the calculation of the slide modulus n when the angle of twist θ due to a measured torque T is observed, ρ and l being known; a weight of known moment of inertia K may be hung by the wire constituting a torsion pendulum, and the constant of torsion $(\pi n \rho^4/2l)$ determined as explained in article 100. Whence, ρ and l being known, n may be calculated.

Proof of equation (79). — Let Fig. 81 represent a sectional view of the rod. Consider the narrow annulus of width Δr and radius r shown by the dotted lines. The force action per unit area across this annulus is $rn\theta/l$ according to equation (78)*b*, and this force action is at right angles to r at each point of the annulus. Therefore the torque action, about the axis of the rod, of the force which acts across unit area of the annulus is $rn\theta/l \times r$; which, multiplied by the area $2\pi r \cdot \Delta r$ of the annulus gives the total torque action ΔT across the annulus. That is

$$\Delta T = \frac{2\pi r^3 n\theta}{l} \cdot \Delta r$$

or

$$T = \frac{2\pi n\theta}{l}\int_0^\rho r^3 dr = \frac{\pi n \rho^4 \theta}{2l}$$

147. Indirect determination of elastic moduli. — The four elastic constants of a substance, namely, stretch modulus $E(=1/a)$, Poisson's ratio b/a, bulk modulus V, and slide modulus n, always satisfy equations (69) and

(72) and therefore any two of these constants being given the other two may be calculated.

148. Limitations of Hooke's law.* — Many substances, such as steel and quartz, conform very closely to Hooke's law for small strains, — stretches of one or two thousandths, — but all substances deviate widely from this law when the strain becomes great. Many substances are more or less plastic, and do not conform to Hooke's law even for small strains.

The behavior of mild steel under increasing longitudinal stress is shown in Fig. 82. The line up to the point p is sensibly straight (strain proportional to stress). Above the point p the line becomes sensibly curved, and a slight *permanent set* remains if the stress is relaxed. A substance strained until it begins to take a permanent set is said to be at its *elastic limit.* The given sample of mild steel is at its elastic limit when the stress is about 60,000 pounds per square inch, and the stretch about 0.0042. The point q at which the stretch suddenly increases and causes a partial

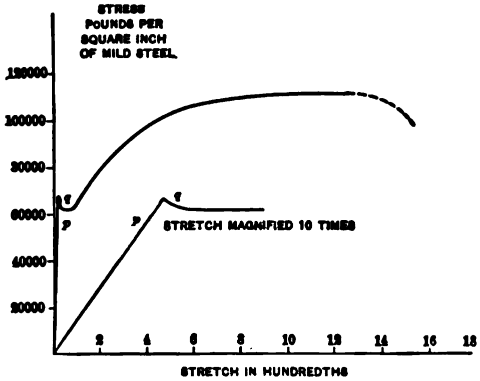

Fig. 82.

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* The behavior of bodies under stress is far too complicated to be fully discussed here. The brief statements given in Arts. 148 to 153 are mainly intended to call the attention of the reader to the fact that Arts. 110 to 147 treat merely of the *geometry* which is necessary to the study of elasticity. For full discussions of the elastic properties of substances see J. B. Johnson, "Materials of Construction," Wiley & Sons, 1897, and article *Elasticity*, Ency. Brit., 9th ed.

relief of the stress is called the *yield point.* After passing the yield point, a large permanent set is left when the stress is relaxed. The maximum stress is called the *breaking stress*, or, in case of longitudinal stress, the *tensile strength* of the material. The given sample of mild steel has an ultimate tensile strength of about 113,000 pounds per square inch.

149. Resilience. — The work done per unit volume in straining a substance to its elastic limit is called the *resilience* of the substance. This work per unit volume is equal to one half the product of the (limit) stress into the (limit) strain (Art. 140), and it is represented by the area under the straight portion of the curve of Fig. 82. The resilience of mild steel is thus $\frac{1}{2} \times 60{,}000 \frac{\text{lbs.}}{(\text{inch})^2} \times 0.0042 = 126 \frac{\text{inch-lbs.}}{(\text{inch})^3} = 10.5 \frac{\text{ft.-lbs.}}{(\text{inch})^3}$(foot-pounds per cubic inch). Thus it would take about 800 lbs. of mild steel made into a spring to store energy sufficient to furnish one horse power for one minute, provided the spring be so designed as to be strained in every part to its elastic limit when wound up. The resilience of spring steel is two or three times as great as that of mild steel. The resilience of cast iron is less than one foot-pound per cubic inch.

The resilience of a substance is a measure of its strength to withstand a sudden shock, inasmuch as a blow of a hammer, for example, bends a bar until the kinetic energy of the hammer is all used in bending the bar. A structure subject to shocks should be made of highly resilient material.

150. Elastic hysteresis.— In nearly all substances there is more or less of a tendency for strain to persist after the stress has ceased. This is of course very markedly the case where a substance is strained beyond the elastic limit, but in many substances the elastic limit is by no means sharply defined and very slight strains do not entirely disappear when the stress ceases. When a substance is subjected to a stress which increases and decreases periodically between two limiting values S_1 and S_2 the relation between stress and strain is somewhat as indicated in

Fig. 83, where ordinates represent stress and abscissas represent strain. The branch *a* of the curve represents the relation between stress and strain while stress is increasing, and the branch *b* represents the relation between stress and strain while stress is decreasing. This divergence of the curve of stress and strain for increasing and decreasing stress is called *elastic hysteresis*. The increasing and decreasing stress is here supposed to increase and decrease very slowly. If the increase and decrease is rapid the divergence of the two curves *a* and *b*, Fig. 83, is due to hysteresis and also to elastic lag.

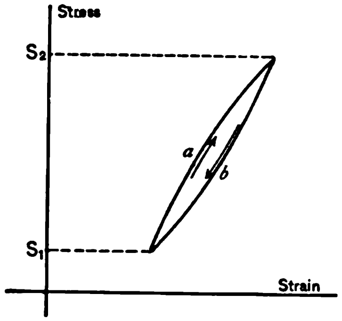

Fig. 83.

151. Elastic lag; viscosity. — Many substances, glass for example, when subjected to stress take on a certain amount of strain quickly, after which the strain increases slowly for a time; and when the stress is relieved, a remnant of the strain persists for a time. This phenomenon is called *elastic lag*.

The strain of some substances continues to increase indefinitely, although it may be very slowly, when they are under stress. Such substances are said to be *viscous*. Nearly all metals are viscous when subjected to great stress.

Elastic hysteresis, elastic lag, and viscosity cause energy to be dissipated in a substance when it is strained. Thus the vibrations of a spring die away rapidly even in a vacuum, on account of the conversion of energy into heat as the spring is repeatedly distorted.

152. Elastic fatigue. — The repeated application of a stress weakens a metal so that it will break under less than its normal breaking stress. Fig. 84 shows the decrease in tensile strength of a sample of mild steel with repetitions of the stress.

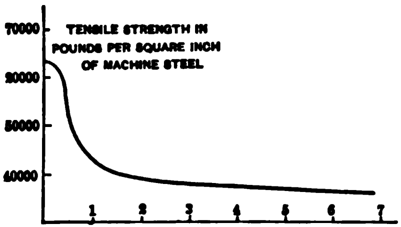

Fig. 84.

Continued repetition of stress causes an increase in the amount of energy dissipated by elastic lag and viscosity. Thus the vibrations of a torsion pendulum die away faster after it has been kept vibrating for several days, than at first.

153. Effects of temperature. — Increase of temperature causes a decrease in the elastic moduli of most substances. Fig. 85 * shows the variation of the stretch modulus of a steel wire with temperature.

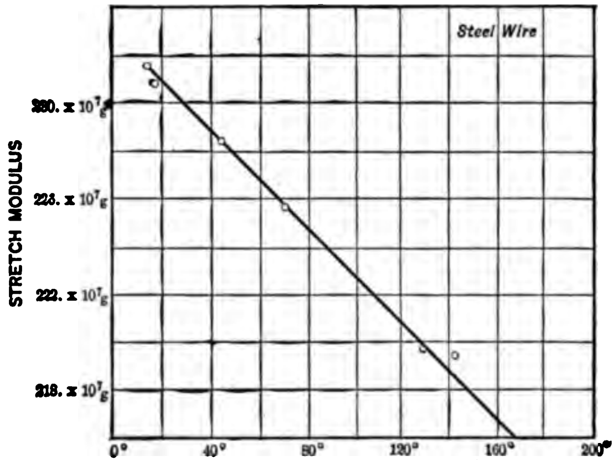

Fig. 85.

154. Friction. — A body in motion is always acted upon by dragging forces which oppose the motion and tend to bring the body to rest. This action is called *friction.* Friction always causes an apparent annihilation of energy and the energy apparently annihilated is converted into heat. This conversion of energy into heat by friction is called the *dissipation of energy* and it is discussed in the chapters on heat.

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* See paper of Miss C. M. Noyes, *Physical Review*, Vol. II., p. 277.

Examples. — The flow of a liquid through a channel or a pipe is opposed by a frictional drag. This type of friction is very briefly discussed in Chapter VII. The motion of a body through a fluid, for example, the motion of a train through the air or the motion of a boat through the water, is opposed by a frictional drag. The motion of a body which slides over another, for example, a block sliding over a smooth floor or a shaft sliding in its bearings as it rotates, is opposed by a frictional drag. The motion of a wheel rolling on a road or on a track is opposed by a frictional drag.

155. Coefficient of sliding friction. — Coulomb found in 1781 that the force T necessary to slide a body over a smooth surface is approximately proportional to the force N which pushes the body against the surface. That is,

$$T = \mu N \tag{80}$$

The proportionality factor μ is called the *coefficient of friction* of the rubbing surfaces, it is nearly independent of the area of contact of the sliding surfaces and of the velocity of sliding.

156. Angle of friction. — Consider two bodies with flat surfaces in contact. Let the force F (Fig. 86), with which they act on each other across the surface of contact, be inclined at an angle ϕ to the normal to that surface. Then $F \cos \phi$ is the normal force N pushing the surfaces together, and $F \sin \phi$ is the tangential force T tending to cause sliding. When the ratio $F \sin \phi$

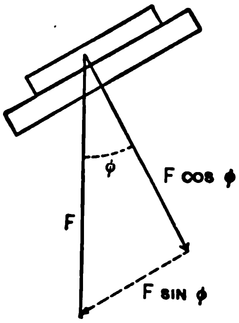

Fig. 86.

$\div F \cos \phi$ becomes equal to the coefficient of friction, μ, of the surfaces, sliding will take place. Therefore the maximum value of the angle ϕ is such that

$$\tan \phi = \mu \tag{81}$$

The angle ϕ is called the *angle of friction* of the two surfaces.

157. Limitations to Coulomb's law of friction.* — The friction between two surfaces is, no doubt, due in part to a continual interlocking and release of fine protuberances on the sliding surfaces and in part to a continual welding together and tearing apart of the smaller parts of the surfaces as they come into intimate contact.

When the sliding surfaces are distinctly rough, there is no regularity whatever in the friction. Surfaces which are comparatively smooth, however, satisfy the law very closely, especially if they are of different materials and are not allowed to rest, for at starting the friction is always greater. Very flat, clean surfaces of the same material, when brought into really close contact, can scarcely be made to slide at all. Glass will not slide on glass if very clean, provided the two faces have been well pressed together, so as to exclude the intervening layer of air. If laid together gently, however, so that an air cushion intervenes, the same surfaces will be almost devoid of friction. Surfaces are lubricated by means of a layer of liquid, and slide with great freedom; the friction is merely that which arises from the viscosity of the lubricant.

158. Rolling friction. — Aside from the sliding friction at the axle bearings, the dragging forces which oppose the motion of a wheeled vehicle are due to what is called *rolling friction.* Rolling friction is very small when a hard wheel rolls upon a hard surface, and is very great when a wheel rolls upon a soft and inelastic surface, as in case of a wagon upon a soft road.

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*Friction belongs to that class of physical actions which involve irreversible thermodynamic processes. In this class of physical actions there is no such thing, strictly speaking, as quantitative correspondence between cause and effect, except on a statistical basis. See discussion of friction by W. S. Franklin, in *Transactions* of American Institute of Electrical Engineers, April, 1903.

CHAPTER VII.

HYDROSTATICS. HYDRAULICS. CAPILLARITY.

FLUIDS AT REST; HYDROSTATICS.

159. Solids and Fluids. — A *solid* is a substance which can sustain a stress which tends to change its shape. One solid resting against another can exert upon it a force which is not normal to the surface of contact.

A *fluid* is a substance which cannot sustain a stress which tends to change its shape. While a fluid is actually changing shape, however, more or less distorting stress must act upon it. Thus a stream of syrup falling from a vessel is under tension like a stretched rope, and the effect of this tension is to continually lengthen each portion of the syrup stream as it falls. A fluid resting against an exposed surface, such as the walls of a containing vessel or the surface of a submerged body, pushes normally against every portion of the exposed surface. A fluid in motion may not push normally against an exposed surface. Thus, a pipe filled with still water is subjected only to a bursting force, while, if the water flows through the pipe, it tends to drag the pipe along with it.*

A *liquid* is a fluid which can have a free surface. A *gas* is a fluid which entirely fills any containing vessel.

160. Pressure at a point in a fluid. — The force F with which a fluid pushes normally against an element of area a of a surface exposed to the fluid *is proportional to the area of the element.* Therefore we may write

$$F = pa \tag{32}$$

in which p is a proportionality factor. This quantity p is called the *hydrostatic pressure* or simply the *pressure* of the fluid *at* the element. In other words the pressure of a fluid at a point is the *force per unit area* with which the fluid will push (normally) on an exposed surface at the point.

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* A liquid jet issuing from the end of a pipe, as is well known, pushes backwards on the pipe. The ultimate seat of this reacting force on the pipe is the normal pressure of the liquid upon the inner walls of the pipe where the pipe bends.

Examples. (a) *Steam pressure.* — The piston of a steam engine is pushed by a force equal to pa, where a is the area of the piston, and p is the pressure of the steam in the cylinder. Every part of the inside surface of a steam boiler is pushed outwards by the steam.

(b) *Atmospheric pressure.* — The force with which the air pushes on the surfaces of bodies does not ordinarily appeal to our senses. It is shown however by the collapse of a thin-walled vessel when the inside pressure is reduced by pumping out the air. Atmospheric pressure is also strikingly shown by means of the apparatus known as the *Magdeburg Hemispheres.* This consists of two metal cups which fit together air tight and form a hollow vessel from which the air may be removed by pumping. The pressure of the outside air then holds the cups very tightly together. This celebrated experiment was devised by Otto von Guerike, the inventor of the air pump. It was performed publicly in Magdeburg in 1654.

(c) *The hydrostatic press* consists essentially of a strong cylinder with a large plunger or piston, and a pump with a small piston or plunger for forcing water into the large cylinder under high pressure. The great forging press at the Bethlehem Steel Works has two plungers each fifty inches in diameter, thus exposing a total of about 3600 square inches of piston area to the water, which is forced into the cylinders of the press at a pressure of 8000 pounds per square inch. This gives a total force of about 14,000 tons upon the two plungers.

161. Pressure in a liquid due to gravity. — The pressure in a fluid under the action of gravity increases with the depth. If the density of the fluid is the same throughout, which is approximately the case with any liquid, then the pressure at a point distant x cm. beneath the surface of the liquid exceeds the pressure at the surface of the liquid by the amount

$$p = xdg \tag{83}$$

in which p is expressed in dynes per square centimeter, d is the density of the liquid in grams per cubic centimeter and g is the acceleration of gravity in centimeters per second per second.

If p is to be expressed in grams' weight per square centimeter then equation (83) becomes:

$$p \text{ in grams' weight per square centimeter } = xd$$

in which x and d are expressed in the same units as before.

Discussion of equation (83). The force with which a liquid acts upon an element of an exposed area is the same whatever the direction of the element of area. This fact was first enunciated by Pascal, and it is known as *Pascal's principle.* By virtue of Pascal's principle we may base our discussion of equation (83) upon the consideration of a horizontal element of area a exposed to the action of the liquid. This element is at a distance x beneath the surface of the liquid so that ax is the total volume of liquid vertically above the element. The mass of this volume of liquid is axd grams, the force with which the earth pulls this mass of liquid is $axdg$ dynes, and this is therefore the total force pushing down on the element so that the force per unit area, or the pressure at the element, is $axdg$ divided by a, or xdg.

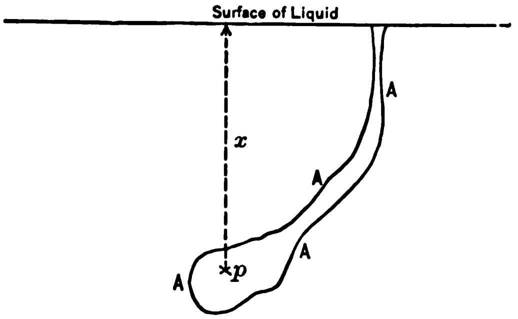

Fig. 87.

Equation (83) involves no consideration of the shape of the vessel which contains the liquid. As a matter of fact the pressure at a point in a liquid is entirely independent of the shape of the containing vessel; it depends only upon the distance x of the point beneath the level of the liquid surface. This may be made almost self-evident from the following considerations: Given a point p, Fig. 87, at a distance x beneath the surface of a large body of liquid. *Imagine a portion of the liquid $AAAA$, of any shape, extending from the point p to the surface.* The liquid surrounding the imagined portion $AAAA$ acts upon $AAAA$ exactly as a containing vessel would act. Therefore the state of affairs at p is the same as it would be if the point p were included in a vessel of any shape whatever, provided only that the vessel is sufficiently tall and filled with liquid to a height x above p.

162. Compressibility of fluids. — Solids and liquids generally decrease but slightly in volume when subjected to increased pressure.* Thus the volume of water decreases but one ten-thousandth part when subjected to a pressure of about 2 atmospheres. The volume of steel decreases but one ten-thousandth part when subjected to a pressure of 147 atmospheres.

On the other hand, gases decrease gently in volume when subjected to increased pressure.

163. The law of Boyle. — *The volume v of a gas,* the temperature of which remains unchanged, *is inversely proportional to the pressure to which the gas is subjected.* That is,

$$v = \frac{C}{p} \tag{84}$$

in which C is the proportionality factor. This equation may be written thus:

$$pv = C \tag{85}$$

This relation was discovered by Boyle in 1662, and more completely established by Mariotte, who discovered it independently in 1676.

164. Experimental verification of Boyle's law. — Air is entrapped in the short arm a of a U-shaped glass tube, TT (Fig.

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* Therefore the bulk modulus of a liquid is large. See Art. 134.

88), the long arm of which is open to the atmosphere. A vessel *m* containing mercury communicates with *TT* by means of a flexible tube; and by raising and lowering this vessel the pressure of the air in *a* may be varied at will. The *volume* of the enclosed gas, as indicated by the length of the tube which it occupies, is found to be inversely proportional to its *pressure*, which is the pressure of the atmosphere (indicated by a barometer) plus the pressure due to the mercury column h_1h_2. Figure 89 shows a convenient form of the apparatus for experimenting with pressures less than atmospheric pressure. A glass tube *t* closed at one end is nearly filled with mercury and inverted in a deep reservoir of mercury. With this apparatus the *pressure* of the entrapped air is the pressure of the atmosphere less the pressure due to the mercury column h_1h_2. The latter may be

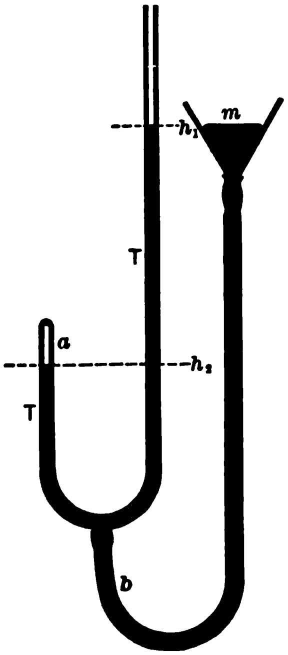

Fig. 88.

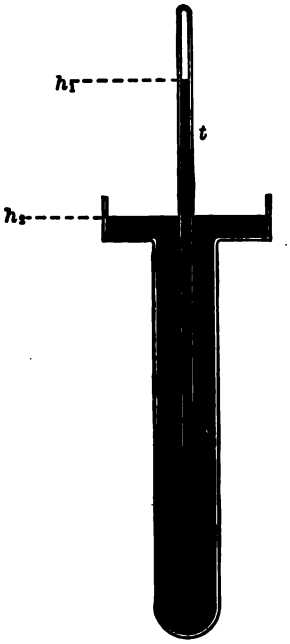

Fig. 89.

varied by raising and lowering the tube, and the variations of volume thus produced may be observed as before.

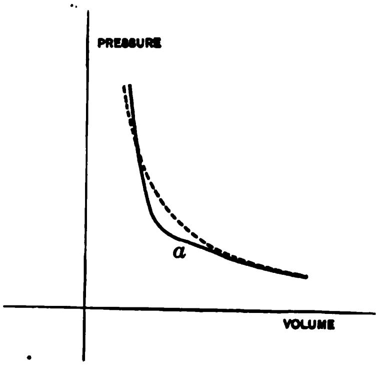

Fig. 90.

165. Limitation of Boyle's law. — The dotted curve (Fig. 90) shows the relation which would subsist between the pressure and volume of a gas if Boyle's law were exactly true, and the heavy line curve shows in a general way the actual relation between pressure and volume of a gas. If the temperature of the gas is lower than a certain critical value, the gas condenses to a liquid when the pressure becomes great enough, and the portion *a* of the curve becomes a horizontal straight line. The discrepancy of the two curves (Fig. 90) for a given gas is greater at low temperatures than at high temperatures, and at a given temperature the discrepancy varies greatly with different gases. At ordinary room temperatures the discrepancy is very small in case of hydrogen, nitrogen, and oxygen; in case of CO_2 it is much greater, and in case of NH_3 it is greater still.

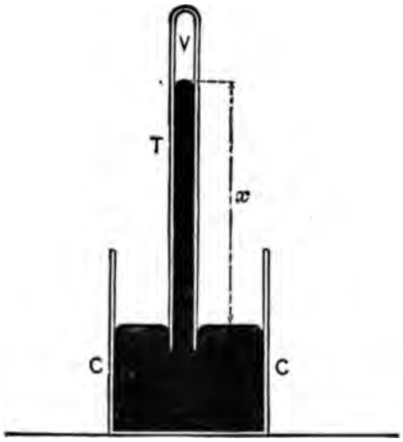

Fig. 91.

166. Measurement of pressure; the barometer. — The barometer consists of a glass tube *T* (Fig. 91) filled with mercury and inverted in a cistern of mercury *C*, the tube being of such length that an empty space *V* is left, in which the pressure is zero. The pressure p at the level of the mercury in the cistern (atmospheric pressure) exceeds the pressure in V by the amount xdg [equation (83)], so that

$$p = xdg. \qquad (83)\text{bis.}$$

In this equation p is the pressure expressed in *dynes per square centimeter*, the height x is measured in centimeters; d is the density of the mercury in grams per cubic centimeter, and $g \frac{\text{cm.}}{\text{sec.}^2}$ is the acceleration of gravity.

If the mercury is at some standard temperature, d will be invariable, and if observations are always taken in one locality, g will be invariable. In this case the pressure is proportional to x, and this distance x may be used as a measure of the pressure. Thus a pressure of 760 millimeters is one which will sustain a column of mercury of that height, the mercury being at standard temperature (0° C.), and the intensity of gravity being $980.61 \frac{\text{cm.}}{\text{sec.}^2}$ (its value at 45° north latitude at sea level).

To facilitate the accurate use of the barometer in different localities and at different temperatures, tables * have been published, with the help of which an observer may find what the value of x (Fig. 91) would be under standard conditions when he has observed its value under known conditions.

167. Measurement of pressure; manometers. — The barometer is used for the measurement of the pressure of the atmosphere, generally for meteorological purposes. An instrument for the measurement of artificial pressure — for example, the pressure in a steam boiler — is called a *manometer*, or a *pressure gauge*.

168. The open tube manometer. — The vessel, a (Fig. 92), in which the pressure to be measured exists, communicates as shown with one arm of a U-tube containing mercury. The other arm of the U-tube is open to the atmosphere. The difference in level, h', is observed. Then the pressure in a is equal to $x + h'$, where x is the pressure of the atmosphere as indicated by a barometer.

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* To be found in most of the reference books for the physical laboratory.

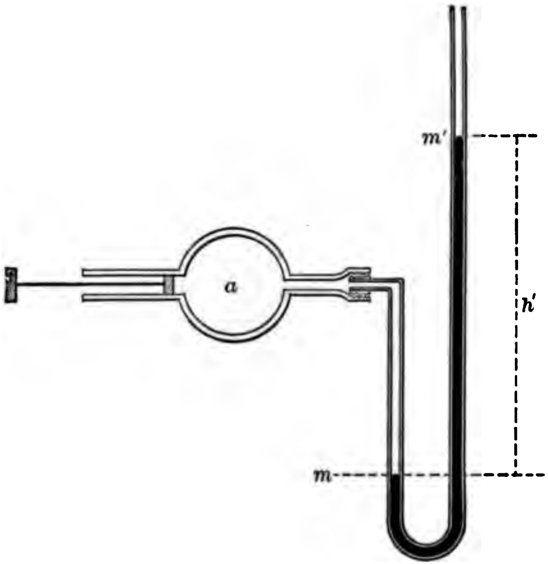

Fig. 92.

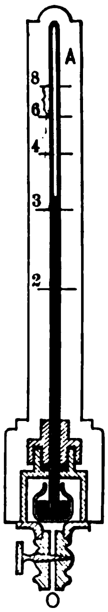

Fig. 93.

169. The closed tube manometer. — A given volume of air at standard atmospheric pressure is, according to Boyle's law, compressed to $\frac{1}{n}$th of its volume when subjected to a pressure of n atmospheres. This principle is employed in the closed tube manometer. Air is entrapped in the part *A* of a glass tube shown in Fig. 93, and its volume, v, at standard atmospheric pressure is marked on the tube at 1. The entrapped air is subjected to the pressure to be measured, by connecting the opening at *O* to the vessel in which the pressure exists, and the pressure is indicated in atmospheres on a scale, the numbers of which are the inverse volumes of the entrapped air.

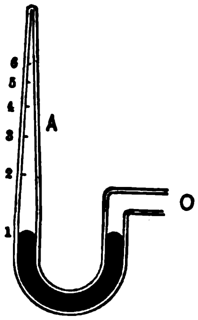

Fig. 94.

Figure 94 shows a closed tube manometer designed to give a scale with marks more nearly equally spaced.

170. The McLeod gauge, for very small pressures.—A glass bulb, *B* Fig. 95, of known capacity carries a capillary tube, *t*, of which the cubic contents per unit length is known. The bulb *B* communicates as shown, with the vessel, *V*. The vessel *V* and bulb *B* contain air at a very low pressure which is to be measured. The vessel of mercury *M* is raised so as to cut off the communication between *B* and *V*, and compress the air in *B* into the capillary tube until its pressure is reduced to one atmosphere (or to a measurable value). Then the initial pressure of the air in *B* is to one atmosphere as its final volume in the capillary tube is to its initial volume in the bulb.

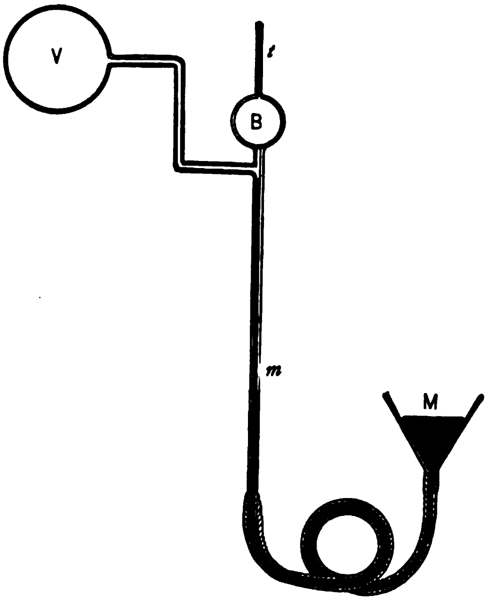

Fig. 95.

171. The gauge tester is a device for the generation of known pressures which are communicated to a pressure gauge which is to be tested. It consists of a small chamber of oil, into which fits a plunger as shown in Fig. 96. The area *a* of the end of the plunger being given, any required pressure, F/a, may be generated by placing weights on the pan as shown; F being the total force pushing on the plunger.

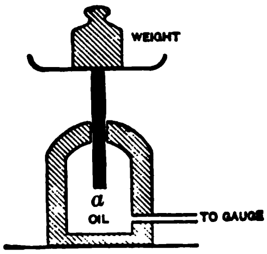

Fig. 96.

172. Archimedes' principle. — A body which is submerged in a fluid is pushed upwards by a force which is equal to the weight of its volume of the fluid. The point of application of this force is called the center of buoyancy; it is the center of mass of the submerged body (considered homogeneous).

Proof. — Consider a *portion* of a fluid at rest. This *portion* is stationary, so that the surrounding fluid is pushing it upwards with a force equal to its weight. The point of application of this force is the center of mass of the *portion*, for that is the only point of application of a force which can support a body against gravity. Now the action of the surrounding fluid upon the *portion* under consideration is the same as upon a body of the same size and shape. Therefore any submerged body is pushed upwards by a force equal to the weight of its volume of the fluid, etc. Q.E.D.

Corollary. — A body which is partly submerged in a liquid is pushed upwards by a force which is equal to the weight of the displaced liquid, and of which the point of application is the center of mass of the submerged part of the body (considered homogeneous).

Remark. — The force with which a fluid acts upon a body wholly or partly submerged is called the *buoyant force* of the fluid upon the body, and its point of application is called the *center of buoyancy.*

173. Examples of buoyant force. — When a body is weighed on a balance, its weight is underestimated by the amount of the buoyant force of the air on the body, and overestimated by the amount of the buoyant force of the air on the weights. The error in the weight of a body due to this cause is sometimes as great as one-half of one per cent.

The buoyant force of the air is easily demonstrated by means of an instrument called the *baroscope*. This consists of a small balance carrying a hollow sphere of glass or metal and a small lead counterpoise, as shown in Fig. 97. The lead counterpoise is adjusted until it is a very little heavier than the sphere in air, and when placed in the exhausted receiver of an air pump the buoyant forces of the air no longer act, and the sphere is seen to be heavier than the counterpoise.

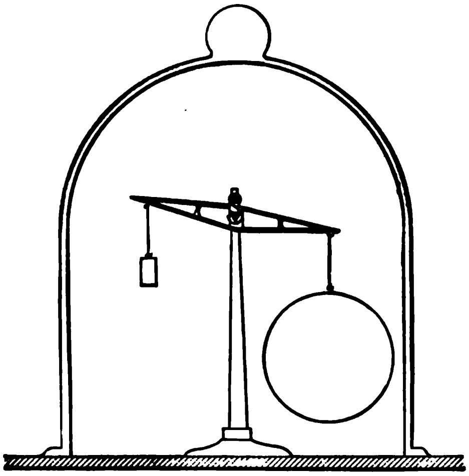

Fig. 97.

174. Floating bodies. — A body floating upon a liquid displaces its weight of the liquid.

When the center of mass g of the body and the center of buoyancy b are in the same vertical line, the body is in equilibrium. That this is the condition of equilibrium is evident, since in general a floating body is acted upon by a torque $gdbe$, Fig. 98, tending whenever the line bg is not vertical to render it so.

175. Stability of equilibrium of a floating body. — The question of the stability or instability of the equilibrium of a floating body is somewhat complicated by the fact that the center of buoyancy shifts with the position of the body. When equilibrium exists, the line *gb* is called the axis. When the body is inclined, the center of buoyancy is shifted to *b′*, Fig. 99.

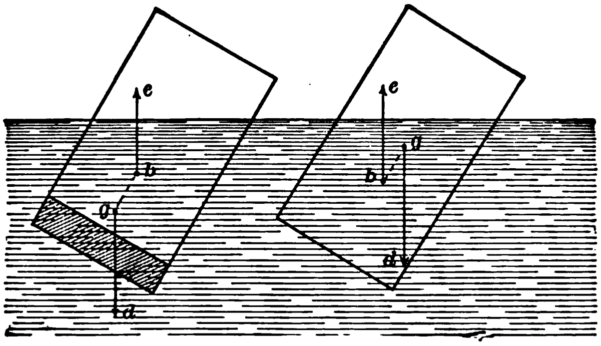

Fig. 98.

A vertical through this new position will cut *bg* in *m*, which point, for the case of an infinitesimal inclination, is called the ***metacenter***. When the metacenter is above the center of mass as in Fig. 99, the equilibrium is

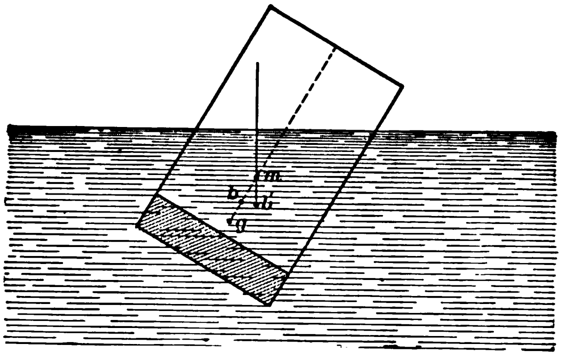

Fig. 99.

stable; when below, it is unstable. Stability is of course a very important matter in the case of a boat.

FLUIDS IN MOTION; HYDRAULICS.

176. Dissipation of energy in a moving fluid. — A fluid in motion dissipates its energy more or less rapidly and comes to rest. A fluid, like syrup, which quickly comes to rest after being stirred is said to be *viscous*. A fluid, like alcohol or water, which continues to move for some time after being stirred is said to be *mobile*.

The cause of the dissipation of energy in a moving fluid is called *fluid friction*. A fluid which would continue to move indefinitely after stirring would be called a *frictionless fluid;* there is no such thing as a frictionless fluid, but the study of the motion of mobile fluids is greatly simplified if we look upon them as (approximately) frictionless. The following articles, 177 to 181, deal with ideal, incompressible, frictionless fluids, and the discussion is of value insofar as the results are approximately applicable to ordinary mobile liquids, as is pointed out in each case.

177. Energy of a Liquid.

(*a*) **Potential energy.** *Proposition.* — The *work per unit volume*, W, required to introduce incompressible fluid into a region under pressure p, is

$$W = p \tag{86}$$

Proof. — Let CC (Fig. 100) be a cylinder provided with a piston of area a, and communicating with a vessel under pressure p. The force F necessary to move the piston is ap (Eq. 82). The work done in pushing the piston a distance Δx is $ap\Delta x$, from (40); but $a\Delta x$ is the volume of liquid forced into the vessel by the movement, so that the work done per unit volume is p.

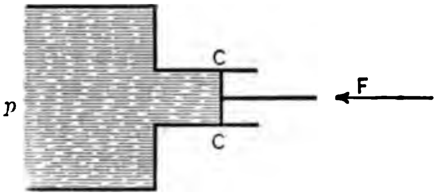

Fig. 100.

Corollary.[*] — The **potential energy per unit volume** of an imcompressible fluid under pressure p is equal to p, for that amount of work must be done to introduce unit volume of the fluid into a region under pressure p.

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[*] This statement, although true in the case in which it is to be used below, requires for generality a number of limiting conditions to be stated. The general aspect of such limitations is as follows: Work *done* is always expressed as the product of two factors, one of which is a generalized *co-ordinate* and the other is a generalized *force*.

(*b*) **Kinetic energy.** — The kinetic energy per unit volume of a moving fluid is

$$W = \frac{1}{2}dv^2 \tag{87}$$

in which d is the density of the fluid, and v is its velocity.

Proof. — Consider a small volume ΔV of the fluid. Its mass is $\Delta V \cdot d$, whence, from equation (44), its kinetic energy is $\frac{1}{2}\Delta V \cdot dv^2$. This divided by ΔV gives $\frac{1}{2}dv^2$ as the kinetic energy per unit volume.

178. Application of the principle of the conservation of energy. — No energy is dissipated during the motion of a frictionless liquid; therefore, *the sum of the potential energy and kinetic energy of such a fluid must remain constant during the motion.*

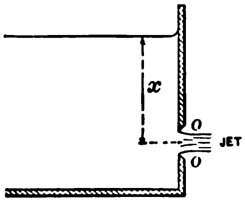

Fig. 101.

179. Efflux. — Consider a tank, Fig. 101, containing a frictionless liquid of density d. Let *oo* be an orifice from which the liquid issues as a jet at a velocity, v, to be determined. Let p be the pressure in the tank at the level of the orifice and p' the outside pressure (atmospheric pressure). In the tank where the velocity of the liquid is inappreciable, the total energy per unit volume is the potential energy p [equation (86)]. In the jet the total energy per unit volume is $p' + \frac{1}{2}dv^2$ [equations (86) and (87)]. As a portion of liquid moves from the tank into the jet its total energy remains unchanged, so that

$$p = p' + \frac{1}{2}dv^2, \tag{88}$$

The *doing* of this work always depends upon the variation of the *generalized co-ordinate.* Thus an incompressible fluid may be brought from zero pressure to any given pressure without the expenditure of *any* work by making the *pressure* the variable.

whence $$v = \sqrt{\frac{2(p - p')}{d}} \tag{89}$$

This equation, expressing the velocity of efflux of a liquid, ignores viscosity and compressibility. The effect of viscosity is to decrease v, and the effect of compressibility is to increase v. For ordinary liquids the effect of viscosity is the greater, and equation (89) gives too large a value for v. For ordinary gases the effect of compressibility is the greater, and equation (89) gives too small a value for v.

If the difference of pressure $p - p'$, Fig. 101, is due to the weight of the liquid in the tank, then $p - p_1 = xdg$ (83), and equation (89) becomes

$$v = \sqrt{2gx} \tag{90}$$

Comparing equation (90) with equation (21), the velocity of efflux is seen to be the velocity which a body would gain in falling freely through the distance x (Torricelli's Theorem).

Remark. — The lines of flow of a liquid through an orifice are somewhat as indicated in Fig. 102. The obliquity of the lines of flow causes the jet to contract after leaving the orifice. When the jet has contracted and become uniform in size, its velocity is uniform and equal to v [equation (89)]. Let a be the sectional area of the contracted portion of the jet, then av is the discharge from the orifice in cubic centimeters per second.

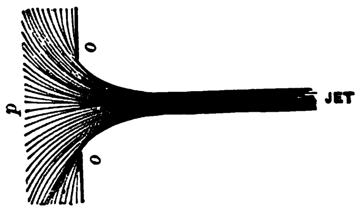

Fig. 102.

180. Variation of velocity of a liquid in a pipe of varying size. — Let a_1 be the sectional area of the bore of a pipe and v_1 the velocity of the liquid in the pipe; then a_1v_1 is the volume of liquid discharged from the pipe per second. If the pipe is of varying size, then the product av (liquid being incompressible) has the same value for all parts of the pipe. Therefore the velocities of a liquid at various points along a pipe are inversely proportional to the sectional areas of the pipe bore at the respective points.

181. Diminution of pressure in a throat. — Consider a pipe aa'' (Fig. 103) through which a frictionless liquid of density d is flowing. Let a and a_1 be the areas of bore of tube, v and v' the velocities of the liquid, and p and p' the pressure of the liquid at a and a' respectively. The total energy per unit volume of the moving liquid at a is $p + \frac{1}{2}dv^2$ and at a' it is $p' + \frac{1}{2}dv'^2$, and since the total energy of a frictionless liquid does not change, we have

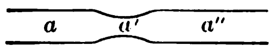

Fig. 103.

$$p + \tfrac{1}{2}dv^2 = p' + \tfrac{1}{2}dv'^2 \tag{91}$$

Also from Art. 180 we have

$$av = a'v' \tag{92}$$

Where the pipe is contracted the velocity of the liquid is large according to equation (92), and equation (91) shows that where the velocity is large the pressure is small. Therefore the pressure in the throat at a' is less than the pressure at a. As the liquid emerges from the throat into the wide portion of the tube at a'' the velocity of the liquid falls off and the pressure rises.

Substituting the value $v' = \frac{av}{a'}$ for v' from (92) in (91), and solving for $p - p'$, we have

$$p - p' = \tfrac{1}{2}d\left(\frac{a^2 - a'^2}{a'^2}\right)v^2 \tag{93}$$

which expresses the diminution of pressure in the throat.

Remark. — It is quite possible for the pressure, p', in the throat to become *negative*, except in case of gases which cannot sustain negative pressure. In case of ordinary liquids, excessive negative pressure breaks them into spray.

The diminution of pressure in a throat is explained directly from Newton's second law of motion as follows: Consider a particle of liquid at a (Fig. 103). This particle gains velocity as it approaches a' and loses velocity again as it approaches a''. Therefore a force must be pushing the particle ahead as it passes from a to a', that is, the pressure behind the particle is greater than the pressure ahead of the particle. Also a force must be opposing the motion of the particle as it passes from a' to a'', that is, the pressure ahead of the particle is greater than the pressure behind it.

Remark. — The volume, per second, of a frictionless fluid discharged from a tank through a tube AB, Fig. 104, *depends only upon the size of the end B of the tube.* In case of viscous fluids, however, the friction is somewhat increased by a contraction at A or between A and B, and therefore the discharge is somewhat lessened by the contraction.

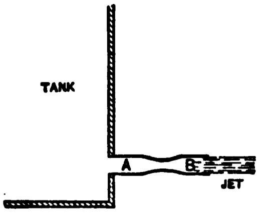

Fig. 104.

Gauging Water.

(*a*) *Orifice meters.* — In these meters the velocity of efflux of water from an orifice of known area is determined by the help of equation (90), the height x being observed, and the discharge is then calculated from the area of the orifice and velocity of efflux. The corrections required on account of the contraction of the jet are previously determined experimentally.

(*b*) *Displacement meters.* — These meters measure water by counting the strokes of a piston in a cylinder, so arranged that the cylinder full of water passes at each stroke.

(*c*) *The Venturi meter.* — This meter depends upon the diminution of pressure in a throat, as follows: Equation (93) may be written in the form:

$$p - p' = \tfrac{1}{2}\, d\left(\frac{a^2 - a'^2}{a^2 a'^2}\right)a^2 v^2 \qquad (94)$$

whence $$av = \left(\frac{2a^2a'^2}{d(a^2 - a'^2)}\right)^{\frac{1}{2}} (p - p')^{\frac{1}{2}} \qquad (95$$

Now av is the discharge per second through the pipe show in Fig. 103, and this can be calculated from equation (9

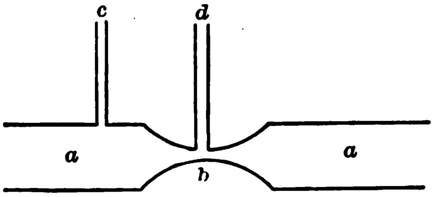

Fig. 105.

when $p - p'$ has been observed and a, a', and d are known. In practice $p - p'$ is indicated by a gauge connected at c and d (Fig. 105).

182. Pressure head of a liquid. — In the above discussion the effects of gravity have been ignored in order that the relation between velocity and pressure in a moving liquid might be brought out in the clearest possible manner. The effect of gravity is to cause an increase of pressure equal to xdg (or a corresponding increase in velocity $\sqrt{2gx}$) whenever in the motion of the liquid it is lowered a distance x.

In many practical problems in hydraulics it is conducive to simplicity to make use of what is called *pressure head* (or simply *head*) instead of *pressure*. The pressure head of a liquid at a given point is the height of liquid column which the pressure at the point can sustain against the pull of gravity. The following articles illustrate the use of the term *head*.

183. Fluid friction. — The effect of friction in the flow of a fluid through a pipe is to cause a loss of head. This loss of head by fluid friction may be demonstrated by the arrangement shown in Fig. 106. Water flows out of a tank through a pipe AB and the pressure or head at each point of the pipe is indicated by the tubes pp in which the water rises as shown.

There are two more or less distinct types of friction of a fluid flowing in a channel or pipe, namely, *eddy friction* and *viscous friction*.* Eddy friction is the loss of head because of the formation of eddies in the flowing fluid.

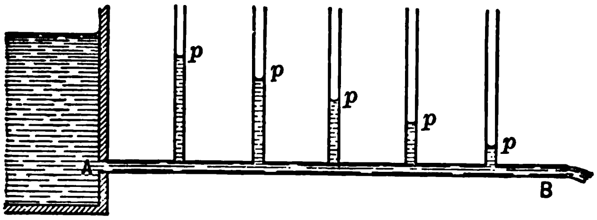

Fig. 106.

184. Loss of head due to eddy friction. — Consider a water pipe AB, Fig. 107, with enlarged portions CCC, through which V cubic centimeters per second of water are discharged. In passing from one chamber to the next the water must attain a velocity $v = \frac{V}{a}$ as it enters the small pipe, a being the sectional area of the bore of the small pipe. This requires a drop in pressure $p - p' = \frac{1}{2} dv^2$. As the water enters the next chamber, it retains its velocity (and therefore the pressure does not rise again), breaks up into eddy streams, and dissipates its energy. Therefore, *the drop in pressure from one chamber to the next, or through any number of chambers, is proportional to* v^2, *or to* V^2. *That is, the loss of head is proportional to the square of the discharge* V.

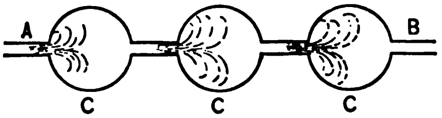

Fig. 107.

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* Theory indicates that eddies could not exist in an ideal frictionless fluid, that is in a fluid without viscosity, so that all fluid friction is ultimately dependent upon viscosity.

This law of eddy friction would be accurately true if the *location* of eddies in a channel were independent of the velocity of the liquid. As a matter of fact however the eddy movements become finer grained with increasing velocity, that is, with increased velocity, a given particle of the liquid gains velocity and loses it again a greater number of times in traveling a given distance; the consequence of this is that the loss of head is proportional to a higher power of the velocity than the square.

In the above discussion the eddies are *localized* by the chambers CCC, Fig. 107.

When water flows through ordinary pipes, eddies occur at elbows, at enlarged portions, and all along the rough walls of the pipe, and the loss of head due to these eddies is nearly proportional to the square of the discharge through the pipe. In all ordinary cases eddy friction is very much larger than viscous friction.

185. Loss of head in a smooth-bore pipe (*viscous friction*). — The loss of pressure p, due to viscous friction in a smooth-bore pipe, is

$$p = \frac{8l\eta V}{\pi\rho^4} \qquad (96)$$

in which l is the length of the tube, ρ is the radius of its bore, V is the volume of liquid discharged per second, and η is a constant called the coefficient of viscosity of the liquid. When ρ is large, the loss of pressure p due to viscous friction is very small. Equation (96) is derived in the following articles in which viscosity is discussed in detail. This discussion involves the elementary theory of stresses and strains. The beginner may get a sufficiently clear knowledge of the meaning of the coefficient of viscosity from the proposition in Art. 187.

186. Stokes' law; coefficient of viscosity. — The action of a shearing stress upon a portion of liquid is to continually lengthen it in the direction of the *pull* and to continually shorten it in the direction of the *push*. That is, ϕ being the angular distortion (angle of shear) of a portion of the fluid, the action of a shearing stress is to cause ϕ to change at a definite *rate*, $\frac{d\phi}{dt}$.

Stokes' law. — Stokes found that the rate, $\frac{d\phi}{dt}$, at which the angular distortion of a portion of liquid changes is approximately proportional to the shearing stress. The constant ratio, shearing stress S, divided by $\frac{d\phi}{dt}$ for a given liquid is called its coefficient of viscosity η, so that

$$\eta = \frac{S}{\frac{d\phi}{dt}} \tag{97}$$

187. Viscous fluid between parallel plates; lubrication. — Let $A'B'$ (Fig. 108) be a plane plate, and AB another, parallel thereto, distant x therefrom, and moving at velocity v, as indicated by the arrow. Let the space between the plates be filled with a viscous fluid. Consider a portion CD of the fluid, which at a given instant is cubical. After the lapse of time Δt, the upper plate will have moved a distance $v\Delta t$, and this cubical portion of fluid will be distorted as indicated by the dotted rhombus, of which the angle of distortion is $\Delta\phi = \frac{v\Delta t}{x}$. Therefore

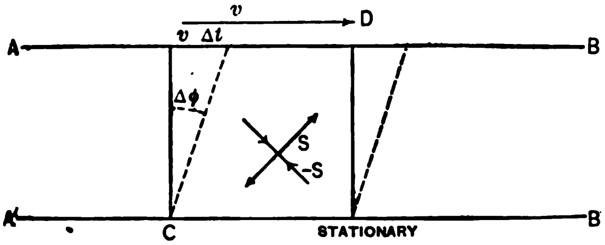

Fig. 108.

$$\frac{d\phi}{dt} = \frac{v}{x} \tag{98}$$

The stress in the liquid, aside from hydrostatic pressure, is a shearing stress of which the two pulls are $+S$ and $-S$ in the directions shown. Substituting $\frac{d\phi}{dt} = \frac{v}{x}$ in (97), we have

$$S = \frac{\eta v}{x} \tag{99}$$

The stress on all horizontal (also on all vertical) sections in the fluid is purely tangential and equal to S. Therefore the plates are acted upon by dragging forces $F = Sa$, in which a is the area of the plate. Substituting the value of S from (99), we have

$$F = \frac{\eta a v}{x} \tag{100}$$

From this equation we have the following:

Proposition. — The force F required to slide a smooth plate of area a over another smooth plate, with a layer of liquid between of thickness x, is proportional to av/x, where v is the velocity of sliding; and the proportionality factor η is called the coefficient of viscosity of the liquid. It is expressed in dynes$\Big/\frac{\text{cm}^2}{\text{sec}}$ in the c.g.s. sytem.

188. Flow of viscous liquid through a smooth-bore tube. — Let ρ be the radius of the bore of the tube, l its length, p the pressure forcing the liquid through, and v the velocity of the liquid at a point distant r from the axis of the tube. Consider a cylindrical portion of the liquid of radius r coaxial with the tube. The surface of this cylindrical portion moves as a solid *rod* through the tube at velocity v. Similarly, the cylindrical surface of radius $r + \Delta r$ moves through the tube as a hollow *shell* at velocity $v + \Delta v$. The fluid layer between this *rod* and *shell* is under conditions similar to the fluid between the plates, as discussed in the previous article. Therefore, writing Δv for v in equation (98), and Δr for x, we have $\frac{d\phi}{dt} = \frac{dv}{dr}$. Substituting this value of $\frac{d\phi}{dt}$ in (97), we have $S = \eta \frac{dv}{dr}$. This stress is tangential over the whole surface, $2\pi rl$, of the *rod*, so that the dragging force on the *rod* is $2\pi rlS$ or $2\pi rl\eta \frac{dv}{dr}$. This dragging force is continually overcome by the force $\pi r^2 p$ pushing on the end of the *rod*, and is equal thereto. We have, therefore, $\pi r^2 p = 2\pi rl\eta \frac{dv}{dr}$, or $\frac{dv}{dr} = \frac{p}{2l\eta} r$, or $v = \frac{pr^2}{4l\eta} + constant$. When $r = \rho$, $v = 0$, so that the constant of integration is equal to $-\frac{p\rho^2}{4l\eta}$. It follows, therefore, that

$$v = \frac{pr^2}{4l\eta} - \frac{p\rho^2}{4l\eta} \tag{101}$$

The velocity at each part of the tube is thus determined. To find the volume V of fluid discharged in time τ, consider a section of the tube (Fig. 109).

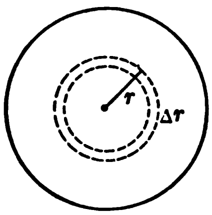

Fig. 109.

The velocity over all the area, $2\pi r\Delta r$, of the dotted annulus, is v, so that the volume ΔV, flowing across this annulus in time τ, is $\Delta V = 2\pi r\Delta r \cdot v \cdot \tau$. Substituting v from (101), we have

$$dV = \frac{\pi p\tau}{2l\eta} r^3 dr - \frac{\pi p\rho^2\tau}{2l\eta} rdr$$

or

$$V = \frac{\pi p\tau}{2l\eta} \int_0^\rho r^3 dr - \frac{\pi p\rho^2\tau}{2l\eta} \int_0^\rho rdr \tag{102}$$

Therefore

$$V = \frac{\pi p\rho^4\tau}{8l\eta} \tag{103}$$

This equation (103) enables the calculation of η when V, p, ρ, τ, and l have been determined

CAPILLARITY.

189. Cohesion; adhesion. — When a body is under stress, as for example a stretched wire, the tendency of the stress is to tear the contiguous parts of the body asunder. The forces which oppose this tendency and hold the contiguous parts of a body together are called the forces of *cohesion*. The forces which cause dissimilar substances to cling together are called the forces of *adhesion*. The discussion of the elastic properties of solids is a discussion of their properties of cohesion. The phenomena exhibited by liquids on account of cohesion are called capillary phenomena; these phenomena are discussed in the following articles.

190. Cohesion of liquids; surface tension. — On account of their cohesion, all liquids behave as if their free surfaces were stretched skins. This is explained in Art. 193. The following experiment* shows the surface tension of water in a very striking manner; it depends upon the fact that a little oil or alcohol greatly reduces the surface tension of water. A thin rubber band is floated on the surface of clean water. This divides the surface into two regions, outside and inside the band. If the inner region is touched with a glass rod which has been previously dipped in oil, its surface tension is diminished, and the

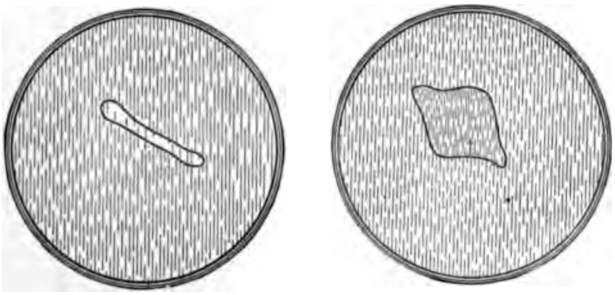

Fig. 110.

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* E. F. Nichols, Physical Review, Vol. I., p. 299.

surface tension of the outer region, as yet unchanged, draws the band into an approach to a circular form as shown in Fig. 110. The surface tension of the outer region may then be diminished in the same way, and the band relapses into its initial shape.

191. Sphere of molecular * action. — Consider a substance, for example a piece of iron. The force necessary to detach a small particle of the iron is the same whether the piece of iron be large or small, thick or thin. That is, a small piece of a substance is as hard to scratch as a large piece. Therefore a given particle of a substance is held only by the particles which are *very near* to it.

A sphere, of which the radius is perhaps not greater than a few millionths of a centimeter, which includes all the particles of a substance which have appreciable action upon the particle at the center of the sphere, is called the *sphere of molecular action* of the substance.

192. Surface film; molecular pressure.—The surface layer of a liquid of thickness equal to the radius of the sphere of molecular action is called the *surface film*. Let AB (Fig. 111) be the plane surface of a liquid. Consider the molecular forces acting upon the fluid particle p. Describe the sphere of molecular action with its center at p. If this sphere lies wholly within the liquid, that is, if p lies below the surface film, the particles which have sensible action on p will be evenly distributed around p and the forces acting on p are the same in all directions. If p lies *in* the surface film, there will be more attracting particles on the inward than on the outward side of p, and there will be on the whole a force pulling p inwards. These inward forces upon the particles of a liquid in the surface film make the hydrostatic pressure in the region enclosed by the surface film greater

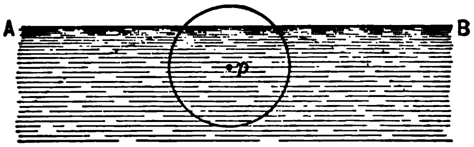

Fig. 111.

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* The word *molecular* is to be understood as referring merely to the *small parts* of a body, and the *action* between two such parts is assumed to be simply an attraction.

than the hydrostatic pressure on the outside. The increase of pressure across the plane surface of a liquid is called the *molecular pressure* of the liquid.

193. Work done by the contraction of the surface of a liquid; surface tension. — When the surface of a liquid contracts, for example, when a drop of liquid changes from an elongated to a spherical form, a number of particles in the surface film move into the interior of the liquid, and work is done by the forces which pull these particles inwards. That is, the contraction of the surface of a liquid means the doing of work by the liquid. The surface film of a liquid is therefore in a state of tension, for any surface which does work upon contraction is in a state of tension.

194. Definition of tension of a surface. — Consider a surface which is in a state of tension, for example a stretched sheet of rubber. The portions of this membrane or film which lie on the two sides of a line AB (Fig. 112) *pull* on each other across the line with a force F which is proportional to the length of the line. The force per unit length of line is the measure of the tension of the film.

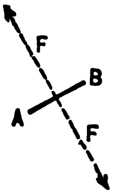

Fig. 112.

195. Angles of contact. — A surface which separates two fluids always makes a definite angle of contact with the clean surface of any solid with which the fluids are in contact. For example, the surface between water and air always meets a clean glass surface tangentially. The surface between mercury and air likewise always meets a clean glass surface at a definite angle. The surfaces which separate three fluids always intersect at definite angles.

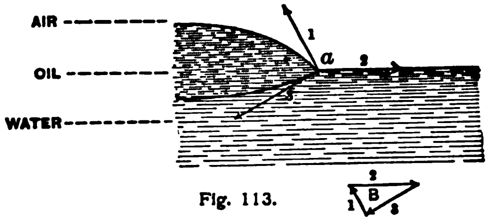

Fig. 113.

Discussion. (*a*) *Case of three fluids.* — Let *a* (Fig. 113) be a line perpendicular to the paper at which three fluid surfaces meet as shown. The forces 1, 2, 3, with which the surface films pull at *a*, are in equilibrium. These forces are therefore parallel, and proportional to the sides of a triangle. This triangle is shown at *B* (Fig. 113). The angles between the surfaces at *a* can thus be calculated in terms of the tensions of the three fluid surfaces.

(*b*) *Case of two fluids and a solid.* — Not only is a surface, separating two fluids, under tension, but so also is a surface separating a solid and a fluid. In this case, however, the surface being rigid, the experimental evidence is less complete. Let *a* (Fig. 114) be the line of contact, with glass, of the surface separating oil and water. The condition of equilibrium requires the resolved part parallel to *AB*, of the tension *S*, to be equal to the difference of the tensions *o* and *w*, so that the angle *θ* is thereby determined.

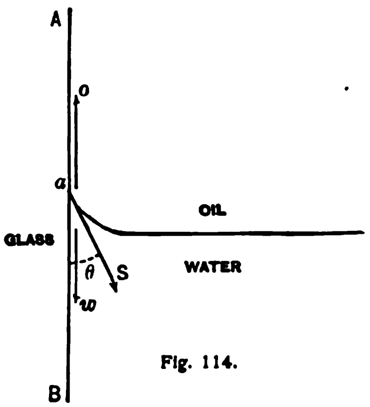

Fig. 114.

196. Capillary elevation and depression. — The angle of contact of a water surface with glass is such as to make the surface of water in a small glass tube concave, as shown in Fig. 115, and the surface tension draws the water up into the tube.

The angle of contact of a mercury surface with glass is such as to make the surface of mercury in a small glass tube convex, as shown in Fig. 116, and the surface tension draws the mercury downwards in the tube.

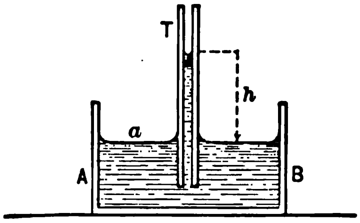

Fig. 115.

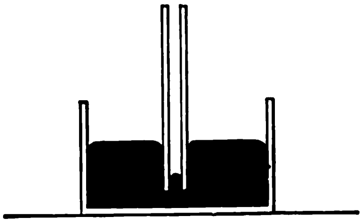

Fig. 116.

197. The law of diameters (Jurin's law). — *The elevation or depression of a liquid in a capillary tube is inversely proportional to the diameter of the tube.*

Jurin's relation between size of capillary tubes and the amount of elevation or depression therein is derived from the foregoing principles as follows: —

Let Fig. 117 represent a capillary tube, bore of radius r, plunged in a broad vessel of liquid. Then $\pi r^2 hdg$ is the weight of the elevated column h of liquid, $\pi r^2 h$ being its volume, d its density, and g the intensity of gravity. The force which sustains this weight is the vertical component, $2\pi rT\cos\theta$, of the force due to the tension of the liquid surface at the walls of the tube, $2\pi r$ being the circumference of the bore of the tube, T the surface tension, and θ the angle of contact. Therefore

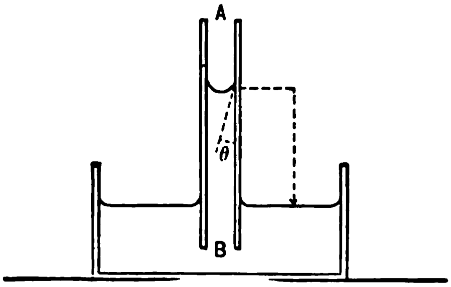

Fig. 117.

$$\pi r^2 hdg = 2\pi rT\cos\theta \qquad (104)$$

whence

$$h = \frac{2T\cos\theta}{dgr} \qquad (105)$$

For water in glass tubes $\theta = 0$, so that for this case (105) becomes

$$h = \frac{2T}{dgr} \qquad (106)$$

This equation enables the calculation of T when h, d, g, and r are known.

198. Equation of capillarity. — The difference in pressure on the two sides of a liquid film, due to the tension of the film, depends upon the tension and upon the curvature of the film The equation expressing this relation is called the **equation of capillarity.** It is

$$p = T\left(\frac{1}{R'} + \frac{1}{R''}\right) \qquad (107)$$

in which p is the difference in pressure on the two sides of a film, due to the tension T of the film, and R' and R'' are the principal radii of curvature of the film at the point.

Proof. — (*a*) The tension of a film is equivalent to two mutually perpendicular equal pulls. The effect of each pull may in any case be considered by itself, the total effect being the sum of the effects due to each.

(*b*) Consider a small portion ΔS of any curved surface in the neighborhood of a chosen point on the surface. Draw a tangent plane to the surface at the point. A plane normal to this tangent plane intersects ΔS in a curve which is sensibly an arc of a circle of radius r. There is one position of this normal plane for which r has a maximum value R', and another position at right angles thereto for which r has a minimum value R''. These two radii, R' and R'', are called the *principal curvatures* of the surface at the point. The effects of each of these two curvatures of a surface at a point may in any case be considered separately. When considering the effect of one curvature the surface may be thought of as cylindrical.

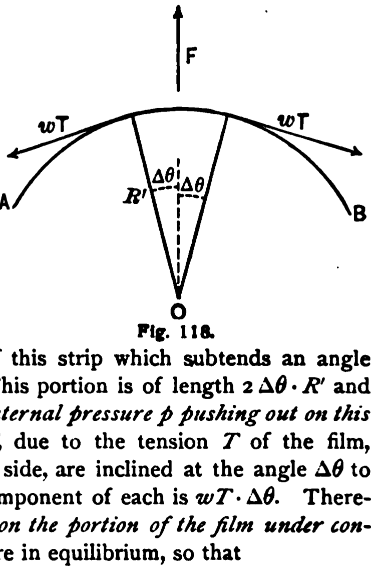

Fig. 118.

(*c*) **Pressure inside a cylindrical film.** — Let AB (Fig. 118) represent a strip, of width w, of a cylindrical film of radius R'. Consider the portion of this strip which subtends an angle $2\,\Delta\theta$ from the axis O of the cylinder. This portion is of length $2\,\Delta\theta \cdot R'$ and of width w; therefore the *force due to internal pressure p pushing out on this portion is $pwR'2\,\Delta\theta$.* The forces wT, due to the tension T of the film, pulling this portion of the film on either side, are inclined at the angle $\Delta\theta$ to the horizontal, so that the downward component of each is $wT \cdot \Delta\theta$. Therefore *the total downward force acting upon the portion of the film under consideration is $2\,wT \cdot \Delta\theta$.* These forces are in equilibrium, so that

$$2\,wT \cdot \Delta\,\theta = pwR'\,2\,\Delta\theta \text{ or } p = \frac{T}{R'} \qquad (108)$$

(*d*) In case of a film having double curvature, with principal radii R' and R'', the internal pressure contributed by each is $\frac{T}{R'}$ and $\frac{T}{R''}$ respectively, so that the total internal pressure is

$$p = \frac{T}{R'} + \frac{T}{R''} \qquad \text{Q.E.D.}$$

199. Examples of the Application of the Equation of Capillarity.

(*a*) **Liquid uninfluenced by gravity.**—In such a liquid the pressure is the same throughout, so that from equation (107) $\frac{1}{R'} + \frac{1}{R''}$ must have the same value at all points of the surface of such a liquid. The only finite surface which satisfies this condition is the sphere. Therefore the surface of a liquid uninfluenced by gravity must be spherical.

The spherical form of a portion of a liquid uninfluenced by gravity is beautifully shown by drops of olive oil floating in a mixture of alcohol and water of the same density. Gravity produces only very slight variations of pressure in a small portion of a liquid, and these always assume a spherical form, as exemplified by drops of dew and globules of mercury, and also by small bubbles of air in a viscous liquid.

(*b*) **Soap bubbles.** — The difference in pressure inside and out of a soap bubble is sensibly of one value at every point of its surface, so that $\frac{1}{R'} + \frac{1}{R''}$ is constant, and the bubble is spherical.

(*c*) **Soap films on wire frames** (Plateau). — In this case the pressure is the same on the two sides of the film, and

$$\frac{1}{R'} + \frac{1}{R''} = 0 \qquad (109)$$

If the wires of the frame lie in a plane, the film itself will be plane, otherwise R' and R'' have equal and opposite values at each point.

CHAPTER VIII.

CHEMICAL ACTION ; CRYSTALLIZATION ; DIFFUSION.

200. Homogeneous substances. — A homogeneous substance is one which is similar in all its parts. Organic tissue, living or dead, is non-homogeneous. Any non-homogeneous substance must be reduced to one or more homogeneous substances before any detailed chemical or physical study can be made of it.

201. Coexistent phases. — It is a matter of common observation that a homogeneous substance may be made to break up into a number of distinct homogeneous parts which differ definitely as to their properties. Such homogeneous parts are called *phases*.

Examples. — A vessel full of steam partly condenses when cooled, and presents two phases, namely, steam and water; further cooling may partly freeze the water, when there will be three phases, namely, water vapor, water, and ice. Water vapor may be wholly converted into water or into ice, and *vice versa;* therefore these three phases are said to be of the *same composition.*

A solution of salt may be made to deposit salt crystals by cooling. The mother-liquor and the crystals are phases of *different composition*, inasmuch as salt solution cannot be wholly converted into salt, and salt cannot be converted into salt solution without admixture of water.

This breaking up of substances into phases of different composition is the fundamental fact of chemistry.

202. Elementary and compound substances. — A substance which can be broken up into phases of different composition is called a *compound substance.* The component parts themselves may be compound. Substances which have never yet been broken up in this way are called *elementary substances* or *chemical elements,* of which about seventy are at present known.

203. The conservation of matter. — The sum of the masses of the component parts of a compound substance is equal to the mass of the compound. That is, the total mass of a number of substances remains constant, irrespective of the changes which these materials may undergo.

Example. — The products of the combustion of coal may be collected and weighed, and the mass of these products is equal to the sum of the masses of air and coal which took part in the combustion.

204. Chemical compounds; mixtures. — Those compound substances of which the component parts are in unalterably fixed proportions* are called *chemical compounds.* Those compound substances of which the component parts may be in varying proportions are called *mixtures.* For example, pure water (which contains always 200 parts by weight of hydrogen to 1588 parts by weight of oxygen) is a chemical compound; air, which may contain nitrogen, oxygen, carbon dioxide, water vapor, etc., in widely varying proportions, is a mixture.

Two substances are said to *combine* when they form a chemical compound. The molecular theory furnishes a useful conception of this act of *combination.*

205. Combining ratios. — Consider two elementary substances. Let a be the mass of the one which combines with a mass b of the other. The ratio $\frac{a}{b}$ is called the *combining ratio* of the two elements.

Law of multiple proportions. — Two elements often have more than one combining ratio. Let $b_1, b_2, b_3, \cdots$, be the masses of one element which can combine with a given mass a of another element. *The masses b_1, b_2, b_3, are always multiples* of some one number, so that the ratios $b_1 : b_2$, $b_1 : b_3$, $\cdots$, are rational fractions.

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* The fact that there are such compound substances is called the *law of constant proportions.*

206. The chemical combination of gases. — Let u and v be the respective volumes of two gases (measured at the same temperature and pressure) which combine. The ratio $\frac{u}{v}$ is always (sensibly) a rational fraction. Thus two volumes of hydrogen combine with one volume of oxygen to form water. Equal volumes of hydrogen and chlorine combine to form HCl. Nitrogen and oxygen may combine in the following proportions by volume: Two of nitrogen with one of oxygen; one of nitrogen with one of oxygen; two of nitrogen with three of oxygen; one of nitrogen with two of oxygen; and two of nitrogen with five of oxygen.

207. The molecular theory.* — The above facts of chemical combination, as well as other experimental facts to be given hereafter, are clearly represented if we assume that each chemical element is made up of *ultimate similar particles of equal mass* called *atoms*, and that the atoms of two or more elements in a chemical compound are arranged in *similar groups* called *molecules*. For example, the atomic groups, or molecules, of the five compounds of oxygen and nitrogen are as follows:

Compound No. 1. Two atoms of nitrogen and one atom of oxygen, N_2O.
Compound No. 2. One atom of nitrogen and one atom of oxygen, NO.
Compound No. 3. Two atoms of nitrogen and three atoms of oxygen, N_2O_3.
Compound No. 4. One atom of nitrogen and two atoms of oxygen, NO_2.
Compound No. 5. Two atoms of nitrogen and five atoms of oxygen, N_2O_5.

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* This molecular theory proves very useful throughout the subject of chemistry, in the theory of gases and solutions, in the theory of heat, in the theory of crystallization, and in the theory of electrolysis. The student should look to this *use*. All elaborations of the theory are for the purpose of furnishing concrete conceptions of obscure chemical and physical phenomena, and until the student takes up the study of such phenomena it is worse than useless to dwell upon the details of the theory.

The combining ratios of these various compounds are 2802 : 1588, 1401 : 1588, 2802 : 4764, 1401 : 3176, and 2802 : 7940 respectively. These numbers are multiples of 1401 on the one hand and of 1588 on the other. The number 1401 is called the *atomic weight* of nitrogen and the number 1588 is called the atomic weight of oxygen, the atomic weight of hydrogen being chosen arbitrarily as 100.

The principle of Avogadro. — The molecular theory taken in conjunction with the facts stated in Arts. 205 and 206 indicates *that all gases have the same number of molecules per unit volume at the same temperature and pressure.*

208. States of aggregation; solids; liquids; gases. — A *solid* is a substance which has more or less rigidity of form or which can withstand, at rest, a shearing stress. A *fluid* is a substance which has no rigidity of form. (See Art. 159.) A *liquid* is a fluid capable of having a free surface and of which the volume is definite. A *gas* is a fluid of which the volume is limited only by that of the closed containing vessel.

The phenomena of heat indicate, as we shall see, that the molecules of all substances are in a state of erratic motion more or less violent. In solids it is very likely that each molecule in its motion does not move to any sensible distance from its mean position, but, on the whole, retains its proper place in the molecular structure (crystalline) of the solid. In liquids and gases each molecule wanders throughout the mass of the fluid, as is shown by the phenomena of diffusion and electrolysis.

209. Crystallization. *Crystalline substances.* — Some liquids (also some gases) under proper conditions, *e.g.* when cooled to a certain temperature, begin to deposit solid parts called *crystals*, which are of regular form and bounded (usually) by plane faces. This phenomenon is called *crystallization.* Solids formed in this way are said to be *crystalline.* The physical properties of crystalline solids, such as *cleavage*, and their elastic and electrical properties, indicate definite molecular *structure.* Indeed, the phenomenon of crystallization is to be thought of as a definite grouping of the molecules as they pass from the freedom of the liquid state to the solid state.

Amorphous substances. — Other liquids under proper conditions, *e.g.* when slowly cooled, become more and more viscous, until to all appearances solid. Such substances have no definite melting point, they give no evidence of molecular structure, and are called *amorphous substances.* Most amorphous solids are very slightly viscous, and exhibit other properties of liquids. For example, glass is known to be an electrolyte, a fact which indicates that some of its molecules or atoms must wander throughout its mass.

Crystalloids; colloids. — Substances which crystallize upon solidification are called *crystalloids.* Some substances, sugar for example, may be either amorphous or crystalline according to the conditions under which the solidification takes place. Some substances, such as glue and albumen, seem to be totally incapable of crystallization. Such substances are called *colloids.*

210. Law of constant angles * (Romé de l'Isle). — During the growth of a crystal the angles between the various faces remain unchanged, and for a given substance these angles are invariable. The amount of deposition on a given face, *i.e.* the growth of that face, varies greatly with circumstances. Therefore the *directions only* of the various faces are significant.

211. Axes of reference; intercepts of a face. — It is convenient to choose as axes of reference three lines parallel respectively to *any three* † *edges of a crystal which do not lie in one plane.*

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* The following discussion of crystallography cannot be followed by the beginner without the help of *crystal models.* Actual crystals are not so good, since the eye scarcely perceives *symmetry of angles* except it be accompanied by *symmetry of dimensions,* which is easily provided in models, but only occurs accidentally in actual crystals. The reader is referred to E. S. and J. D. Dana, A Text-book of Mineralogy, Wiley and Sons, 1886, for a full discussion of crystallography.

† It is evident that these axes may be other than rectangular. The choice of axes of reference here indicated greatly simplifies the statement of the law of rational indices. (See Art. 213.)

Let the plane of a crystal face be extended until it cuts these X, Y, Z axes of reference at distances α, β, and γ respectively from the origin. These distances α, β, and γ are called the *intercepts* of the face.

212. Axial parameters of a crystal. *Indices of a face.* — It is convenient in crystallography to express the intercepts α, β, and γ of a face in terms of three chosen* lengths a, b, c in the directions of the respective axes of reference, such that $\alpha = la$, $\beta = mb$, and $\gamma = nc$.

The chosen lengths a, b, and c are called the *axial parameters* of the crystal, and l, m, and n are called the *indices* of the given face. Since the direction, only, of a face is significant, the intercepts la, mb, and nc may be divided by any number, say a; this is equivalent to displacing the face parallel to itself; the intercepts then become l, $m \cdot \frac{b}{a}$, and $n \cdot \frac{c}{a}$. The permissible values of the indices l, m, and n for the various faces of a crystal are determined from the *law of symmetry* (Art. 215) and the *law of rational indices* (Art. 213), as we shall see, so that a *given crystal is completely characterized by the values of* $\frac{b}{a}$ *and* $\frac{c}{a}$, *the angles between the axes of reference, and by its degree of symmetry*. The values of $\frac{b}{a}$ and $\frac{c}{a}$, and the angles between the axes, are called the *elements of a crystal*, and the degree of symmetry (there are six degrees) determines to which one of six *systems* the crystal belongs.

213. Law of rational indices (Malus). — It is always possible to choose $\frac{b}{a}$ and $\frac{c}{a}$ (Art. 212) for a given crystal so that for every face of the crystal the indices l, m, and n are integers.

The intercepts of a number of faces of a given crystal upon the axes of reference being determined by observation, the numerical values of $\frac{b}{a}$ and $\frac{c}{a}$ are determined upon the condition † that l, m, and n shall be integers for every face. There is a series of values of $\frac{b}{a}$ and $\frac{c}{a}$ which satisfy this condition, the smallest values being chosen.

214. The molecular theory of crystallization. — All the experimental facts concerning crystals are clearly represented if we imagine a crystal to be built up of molecules which are deposited in *layers* upon the various faces. The molecules in each layer being arranged in *rows*, each row being the line of intersection of two possible faces of the crystal.

The **axes of reference** are thus in the direction of *rows* of molecules. If the spacing of the molecules in these axial rows is imagined to be proportional to the axial parameters a, b, c of the crystal, then the indices l, m, n of any face (or layer of molecules) must be integers. The most familiar example of this, in two dimensions, is afforded by a field of Indian corn in which the hills are in rows both ways. The hills then form diagonal rows. Choose any two rows as axes of reference. Then any other row not passing through the origin has *intercepts* on these axes and these intercepts are integral multiples of the distances between adjacent hills in the respective axial rows.

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* For manner of choice see Art. 213.

† The possibility of this condition in every case is Malus' discovery.

215. Symmetry of crystals. — Whenever any two edges of a crystal are at right angles to each other, or whenever any three edges are mutually rectangular, it is conducive to simplicity to choose such edges as axes of reference.

Crystals are classified according to their degree of symmetry, into six systems, as follows:

Orthogonal systems, to which all crystals belong, which have, possibly,[*] three mutually perpendicular edges, which are chosen as axes of reference. There are three such systems:

(*a*) *The cubic* or *isometric system.* In this system the faces are similarly grouped about each of the three orthogonal axes. Therefore $a = b = c$ (Art. 212), and these crystals are symmetrical to *nine* planes, of which three are the axial planes xy, yz, and zx, and the other six are diagonal thereto.

(*b*) *The tetragonal* or *dimetric system.* In this system the faces are similarly grouped with respect to each of two only of the orthogonal axes. Therefore $a \lessgtr b = c$, and these crystals are symmetrical to *five* planes, of which three are the axial planes and the other two are diagonal planes intersecting along the axis a.

(*c*) *The orthorhombic* or *trimetric system.* In this system the faces are dissimilarly grouped about the three orthogonal axes. Therefore a, b, and c are all different, and these crystals are symmetrical to *three* planes, which are the axial planes.

Inclined systems, to which all crystals, except hexagonal crystals, belong which do not have, possibly, three mutually perpendicular edges. There are two such systems:

(*d*) *Monoclinic system.* Crystals belonging to this system have, possibly, one edge which is at right angles to two others. In this system a, b, and c are unequal.[†] Monoclinic crystals are symmetrical to *one* plane which is the plane of the oblique axes.

(*e*) *Triclinic system.* Crystals belonging to this system have no two possible edges at right angles. Therefore the axes of these crystals are all oblique, and a, b, and c are unequal.* Triclinic crystals have no plane of symmetry; the only symmetry they possess consists in the fact that diametrically opposite parts are similar.

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[*] An edge or a face may be possible on a crystal of a given substance without actually occurring. That is, certain layers of molecules in a crystal may not always show as faces.

[†] A crystal admitting of monoclinic axes, but of which the arrangement of the faces requires two or more of a, b, and c to be equal, falls into one of the previously mentioned systems.

(*f*) *The hexagonal system.* Crystals which have possibly three edges 60° apart in one plane, and another edge at right angles thereto, and of which the faces are similarly grouped with respect to the three coplanar edges, are classified together as *hexagonal* crystals. Such crystals have four axes. The three coplanar axes are called the *lateral* axes. The *lateral* axial parameters are equal.†

There are two subdivisions of *hexagonal* crystals:

(1) Those which are symmetrical to seven planes; namely, the plane of the lateral axes, which is called the *basal plane*, and six planes perpendicular to the basal plane, three of which include the lateral axes and three, called the diagonal planes, midway between them.

(2) Those which are symmetrical to four planes; namely, the basal plane and the diagonal planes just mentioned.

216. Solution. — Some solids when placed in a liquid, for example, sugar in water, are disaggregated; the molecules ‡ of the solid become free and wander about through the liquid. The solid is said to *dissolve* in the liquid, and the mixture, which soon becomes homogeneous, is called a *solution*. Gases are, in many cases, absorbed by a liquid in indefinite proportions, and one liquid often mixes with another. Such mixtures are also called solutions.

Solubility. — When a liquid at a given temperature has dissolved the greatest possible amount of a substance, the resulting solution is said to be *saturated*. The mass of a substance per unit mass of liquid in a saturated solution at a given temperature is called the *solubility* of the substance at that temperature. The solubility of solids in nearly every case increases rapidly with rise of temperature; the solubility of a gas ordinarily decreases with rise of temperature.

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* A crystal admitting of triclinic axes, but of which the arrangement of the faces requires two or more of a, b, and c to be equal, falls into one of the previously mentioned systems.

† On account of the similar grouping of the faces with respect thereto.

‡ The student should interpret the word *molecule* simply as *small part*, except when forming a conception of a chemical compound, in which case a molecule is a definite group of atoms.

Supersaturation. — If a saturated solution of a substance be cooled slowly, the substance is ordinarily slowly deposited, usually as crystals. The beginning of this action requires a nucleus such as a particle of dust, a sharp point of a submerged body, or a crystal of the substance. Such nucleus being absent, the solution becomes *supersaturated* as it is cooled. A supersaturated solution is in a state of unstable equilibrium and in many cases a very slight disturbance of any kind upsets it and causes the deposition of crystals. Liquids which are on the point of freezing behave as saturated solutions.

217. Diffusion. — When two liquids which are miscible, for example, pure water and an aqueous solution of any salt, are brought carefully together so as not to mix by convection, it is found that in the course of time they become uniformly mixed. This phenomenon is called *diffusion;* it shows that the molecules of liquids wander as mentioned in Art. 208. Gases mix in the same way, but with greater rapidity.

Coefficient of diffusion. — The rate of diffusion of a substance in solution is proportional to the concentration gradient of the solution (Graham's Law). *Explanation:* Consider a vessel containing water, with crystals of soluble salt at the bottom. This salt will dissolve, forming a solution which grows less concentrated towards the top of the vessel. The concentration is reckoned as grams of salt per gram of solution. Consider the diffusion across an imaginary horizontal plane area a. Let Δc be the difference in concentration at two points near the plane, distant Δx the one above the other, then $\frac{\Delta c}{\Delta x}$ or $\frac{dc}{dx}$ is called the *concentration gradient at a.* By rate of diffusion is meant the number of grams of salt which diffuse across unit area per unit time. Graham found this rate to be approximately proportional to $\frac{dc}{dx}$. The proportionality constant is called the *coefficient of diffusion* of the salt in the given solvent.

Diffusion through membranes; osmosis. — When two liquids are separated by a porous membrane, it is found that one of them — perhaps the one which wets the membrane with the greater facility — diffuses through the membrane faster than the other. This phenomenon is called *osmosis.* If the liquids are two different aqueous solutions, the salts, being different, pass through the membrane with unequal facility. Colloids pass through membranes very slowly and crystalloids with much greater rapidity.

218. Semi-permeable membranes; osmotic pressure. — Some membranes allow water to diffuse through them quite freely, while salts in solution pass with difficulty. Such are called *semi-permeable* membranes. If a solution of a salt be separated from pure water by a semi-permeable membrane, water will pass through the solution until the pressure of the solution exceeds that of the water by a certain amount. This excess of pressure is called the *osmotic pressure* of the solution. The conception of osmotic pressure plays an important part in the theory of solutions.

Remark. — We shall return to the discussion of solutions and of osmotic pressure in appendix A on the applications of thermodynamics.

HEAT.

CHAPTER IX.

TEMPERATURE. THERMAL EXPANSION.

219. Thermal equilibrium; temperature. When one or more substances are shielded from all outside disturbing influences they settle to a quiescent state in which there is no tendency to further change of any kind. This quiescent state is called the state of *thermal equilibrium.* For example, the various objects in a closed room settle to thermal equilibrium; when a hot stone is thrown into a pail of water the mixture, at first turbulent, becomes more and more quiet and finally reaches a state of thermal equilibrium.

A number of bodies which have settled to a common state of thermal equilibrium are said to have the same *temperature.* When a substance is hot its temperature is said to be high, when cold its temperature is said to be low.

220. The dissipation of energy. *Preliminary statement of the first law of thermodynamics.* — In every actual case of motion the moving bodies are subject to friction and to collision, their energy is rapidly dissipated, and they come to rest. *This dissipation* * *of energy is always accompanied by the generation of heat, and experience shows that the amount of heat generated is equivalent to the energy dissipated.* This fact is called the first law of thermodynamics.

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* The *dissipation* of energy is the conversion of mechanical energy into heat. This is a *degeneration* of energy from a form which is wholly available for the doing of work to a form which is only in part available.

Example of the dissipation of energy. — An engine driving machinery does work continually. The work thus continually generated is not stored in the machinery as potential energy nor as kinetic energy, but is dissipated, and appears as heat at the various places where friction occurs.

221. The molecular theory of heat. — In considering the phenomena of motion we habitually ignore a very important class of movements and of force actions, namely, the relative motion of, and the mutual force action between, the very small parts of bodies, or, briefly, molecular motion and molecular forces. The dissipation of the energy of a system may thus be due to an increase of the energy of the ignored motion. Indeed, it is helpful to think of heat as *the energy stored in a substance by virtue of the state of its molecular motion.**

222. Effects of heat. — When a substance is heated, by the dissipation of energy, by contact with a hot body, or otherwise, the following effects may be produced:

(1) The temperature of the substance may rise.

(2) The substance may expand or contract.

(3) The substance may melt or vaporize.

(4) If the substance is a chemical compound it may be dissociated. For example, water vapor breaks up into a mixture of oxygen and hydrogen when sufficiently heated.

(5) The substance may exhibit certain electrical phenomena.

223. Thermal Expansion of Gases; Gay Lussac's Law.

(*a*) When equal volumes of various gases are heated, under constant pressure, they all suffer the same expansion for the same rise in temperature.

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* Heat is to be thought of as in part the potential energy of molecular configuration, and in part the kinetic energy of molecular motion. The use of this conception of heat enables the student to grasp some of the facts concerning heat more easily than would otherwise be possible. It is, therefore, used throughout the succeeding chapters, especially in the discussion of the second law of thermodynamics.

This fact was discovered by Gay Lussac, and afterwards by Charles. It may be easily verified by means of the apparatus shown in Fig. 119. This apparatus consists of a glass bulb which contains the gas to be studied. The gas is separated from the air by mercury, which fills the horizontal stem and allows the gas to expand without increase of pressure.

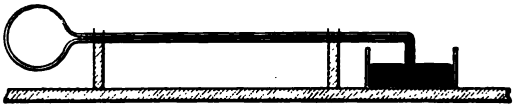

Fig. 119.

(*b*) When various gases under the same initial pressure are heated, and not allowed to expand, they all suffer the same increase of pressure for the same rise in temperature.

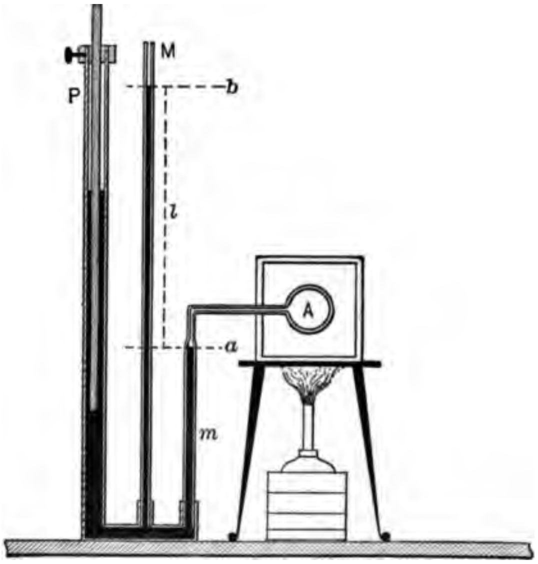

Fig. 120.

This fact is a necessary consequence of Boyle's law (Art. 163) and Gay Lussac's law as stated under (*a*). It may be easily verified by means of the apparatus shown in Fig. 120. The glass bulb A contains the gas to be studied. The bulb communicates, as shown, with the short arm of an open tube manometer Mm, which serves to measure the pressure of the gas in A. P is a tube containing mercury, in which an iron rod may be submerged to any required extent, so as to bring the mercury in m to a marked point at a, thus keeping the volume of the enclosed gas constant.

Remark. — When first discovered, Gay Lussac's law was thought to be exactly true. Very careful measurements, however, show perceptible differences of expansion of various gases. Such gases as carbon dioxide, ammonia, and water vapor, for example, are distinctly different in their expansion from such gases as hydrogen, nitrogen, and oxygen.

224. Provisional definition of the ratio of two temperatures. — Gay Lussac's law affords a convenient basis for the definition of temperature ratios — convenient because not dependent upon any particular gas. The ratio of two temperatures (provisionally defined) is the ratio of the pressures of a constant volume of gas at the respective temperatures. That is, p and p' being the pressures of a constant volume of gas at temperatures T and T' respectively, we have by definition

$$\frac{T}{T'}=\frac{p}{p'} \tag{110}$$

This provisional definition will be found to coincide with the thermodynamic definition of temperature.

225. The air thermometer.* — The air thermometer is an apparatus for measuring the ratio of two temperatures by observing the pressures of a confined portion of gas at the respective temperatures. The essential parts of the air thermometer are shown in Fig. 121. The bulb A contains dry air, and communicates through a tube of fine bore with the short arm of a syphon barometer *BB* (or open tube manometer), by means of which the pressure of the air in the bulb is measured. A movable reservoir *R* containing mercury communicates with the barometer through a flexible tube, and serves to bring the surface of the mercury at *a* to a marked point near the opening of the fine tube, thus keeping the volume of the enclosed air constant.

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* The hydrogen thermometer is the accepted standard.

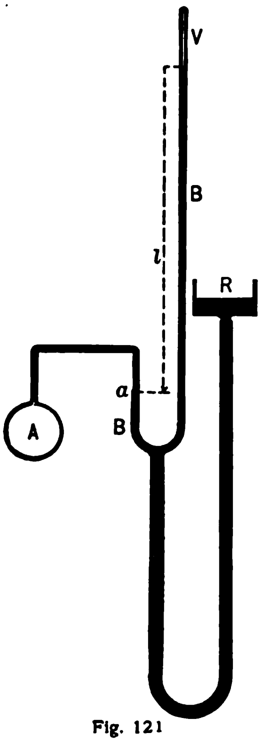

Fig. 121

Remark. — Since the air thermometer measures only the ratio of two temperatures, it is necessary to assign an arbitrary value to some standard temperature, or to the difference between two standard temperatures before numerical values can be found for other temperatures.

226. Standard temperatures. — Experiment shows that the temperature of pure melting ice and the minimum temperature of pure steam at a given pressure are invariable. These temperatures (at 76 c.m. pressure) are taken as the standard temperatures in thermometry. They are called the ice point (*I*) and the steam point (*S*) respectively.

The ratio of standard steam temperature to ice temperature has been found by the air thermometer to be 1.367; that is,

$$\frac{S}{I} = 1.367 \qquad (111)$$

It is customary to assign the arbitrary value 100 to the difference $S - I$, so that

$$S - I = 100 \qquad (112)$$

From these two equations we find that $S = 373$ and $I = 273$ approximately. The values of *S* and *I* are thus known, and any other temperature may be determined by measuring its ratio to I or to S by means of the air thermometer. Temperatures measured in this way are called *absolute temperatures.* Air-thermometer temperatures reckoned from ice point, that is $(T-I)$, are very nearly the same as the temperature values indicated by the ordinary centigrade mercury-in-glass thermometer, which is described in Art. 227.

227. The mercury thermometer. — In nearly all measurements of temperature it is much more convenient to observe the expansion of a liquid contained in a closed vessel of glass, than to use an air thermometer. The most important instrument of this class, and the one universally employed in scientific work, is the mercury thermometer. A glass tube AB (Fig. 122) of fine uniform bore, having a bulb on one end, is completely filled with mercury at a temperature somewhat above the steam point, and is hermetically sealed at A. As it cools, the mercury contracts more rapidly than the glass, and thus only partially fills the stem. The whole is placed in an ice bath, and the position of the surface of the mercury in the stem is marked at I. In a similar manner the steam point is marked at S.

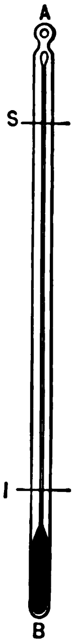

Fig. 122.

In the **centigrade scale (Celsius)**, which is the scale universally used in scientific work, the distance SI is subdivided into one hundred equal parts, which divisions are continued above S and below I. These marks are numbered upwards, beginning at I, which is number zero. The marks below I are numbered negatively from I.*

Any temperature is specified by giving the number of the mark at which the mercury stands when the thermometer is brought to that temperature. For example, 65° C., read *sixty-*

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* The only other scale of which mention need be made is that of Fahrenheit, in which the distance SI is subdivided into one hundred and eighty equal parts, which divisions are continued above S and below I. These marks are numbered upwards, beginning with the thirty-second mark below I, which is number zero. The marks below zero are numbered negatively.

five degrees centigrade, is the temperature at which the mercury in a mercury thermometer stands at mark number 65 of the centigrade scale.

228. Errors of thermometers.[*] *Mercury-in-glass temperatures.* — The indications of two mercury thermometers may differ: (*a*) because of irregularities in the bore of the stem; (*b*) because of errors in the location of the steam and ice points; (*c*) because of irregularities in the divisions on the stem; and (*d*) because of the use of glass having other than standard composition. Also the indication of a given thermometer changes slowly for an hour or more after it is brought to a given temperature. Thermometers used in accurate work are made of a standard variety of glass, the errors due to (*a*), (*b*), and (*c*) are determined as explained in the next article, and readings are taken after the thermometer has stood for some time at the temperature to be indicated. Temperature values indicated by a mercury-in-glass thermometer used in this way are called *mercury-in-glass temperatures*. Mercury-in-glass temperatures are slightly different from air-thermometer temperatures (reckoned from ice point) because of irregularities in the expansion of mercury and glass, as shown in the following table:

TABLE.

AIR-THERMOMETER TEMPERATURES RECKONED FROM ICE POINT.		MERCURY-IN-GLASS TEMPERATURES.
− 19.85		− 20.
0.00	ice point	0.
+ 19.92		+ 20.
39.89		40.
59.90		60.
79.95		80.
100.00	steam point	100.
120.05		120.
140.09		140.
160.10		160.
180.06		180.
199.96		200.

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[*] In accurate work the whole thermometer, stem and all, is brought to the temperature to be indicated; or allowance is made for the difference in temperature of bulb and stem.

229. Standardization of a Mercury Thermometer.

Preliminary. — Consider two mercury-in-glass temperatures t and t'. Let v and v' be the volumes of the bore of the stem from I (ice point) to t, and I to t' respectively; then

$$\frac{t}{t'}=\frac{v}{v'} \qquad (113)$$

This is evident when we consider that t and t' are the numbers of equal divisions of a tube of uniform bore from I to t, and from I to t' respectively.

Method of standardizing. — In the following discussion the thermometer scale is considered to be simply a scale of parts.

(*a*) Observational data. (1) The scale reading a (Fig. 123), when the thermometer is in an ice bath, is taken. (2) The scale reading f, when the thermometer is in a steam bath at the observed pressure of p m.m., is taken. The temperature, t, of a steam bath at pressure p m.m., provided p is not very different from 760 m.m., is

$$t = 100^\circ - 0.0375\,(760 - p) \qquad (114)$$

This equation results from the observed fact that the temperature of a steam bath in the neighborhood of 760 m.m. pressure decreases 0.0375 degree centigrade for each millimeter decrease of pressure. (3) Let V be the volume of the bore between a and f. Let a portion, of volume nearly equal to $\frac{1}{4}V$, of the mercury filament be detached, so that placed end to end four times it may very nearly reach from a to f. Let this filament be placed with its lower end at a, and let the position b of its upper end be noted. Then let it be placed with its lower end at b, and let its upper end be read, and so on. The readings thus obtained are b, c, d, and e.

(*b*) Calculation of true mercury-in-glass temperature R corresponding to a given reading r. Assume the bore of the tube to be uniform throughout each section ab, bc, cd, and df. Let x be the volume of the detached filament; then

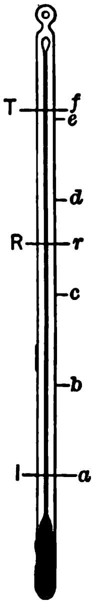

Fig. 123.

$$V = 4x + \frac{f-e}{e-d}\,x \qquad (115)$$

for, since the bore is assumed uniform between d and f, the volume of the portion ef of the bore is $\frac{f-e}{e-d}\,x$.

Similarly, the volume of the bore between a and r is

$$V' = 2x + \frac{r-c}{d-c}\,x \qquad (116)$$

From equation (113) we have $\frac{t}{R}=\frac{V}{V'}$, whence, substituting values of V and V' from (115) and (116) and solving for R, we have

$$R = \frac{2+\dfrac{r-c}{d-c}}{4+\dfrac{f-e}{e-d}}\,t \qquad (117)$$

This formula differs according to the position of the reading r. The assumed uniformity of the bore of the stem throughout the sections *ab*, *bc*, etc., leaves slight outstanding errors; these are, however, very much smaller than would result from an assumed uniformity of bore throughout. Such outstanding errors may be in turn eliminated by calibrating the respective sections.

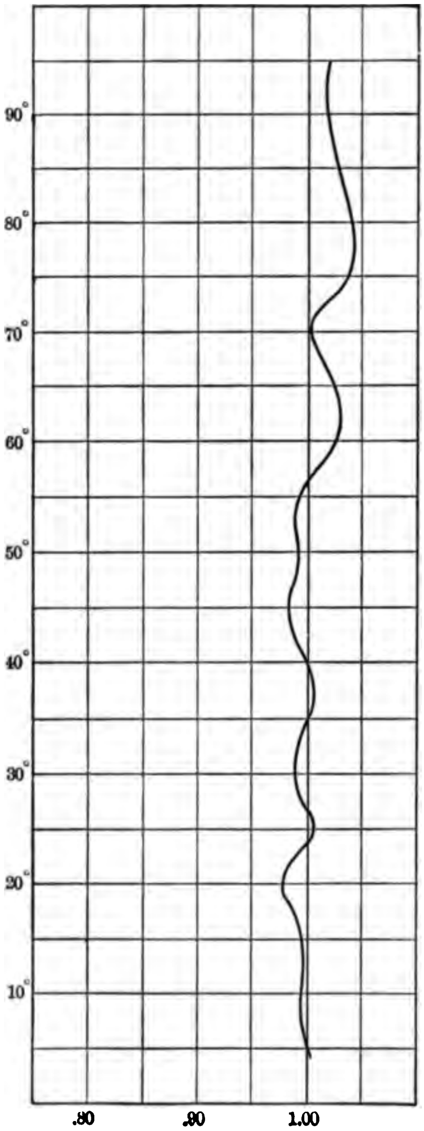

Fig. 124.

How important such calibrations of the tubes of mercury thermometers really are, will be seen from Fig. 124, in which are shown graphically the irregularities of the bore of an excellent thermometer between 5° C. and 95° C. Ordinates are distances from the ice point, and abscissas are cross-sections of the bore, in terms of that of the average cross-section taken as unity. The ratio of the largest cross-section measured, to the smallest was $\frac{1.042}{.976} = 1.068$ (nearly 7 per cent).

230. Thermal expansion of liquids and solids. —

Liquids and solids do not show the regularity of expansion which is found in the case of gases, but every solid and every liquid exhibits characteristic irregularities. Most solids and liquids expand with rise of temperature.

Coefficient of linear expansion. — Let l_0 be the length of a bar of a given substance at 0° C. and l_t its length at t° C. The increment of length $l_t - l_0$ is proportional to the initial length, and approximately proportional* to the rise in temperature t. That is,

$$l_t - l_0 = \alpha l_0 t \qquad (118)$$

or

$$l_t = l_0(1 + \alpha t) \qquad (119)$$

The proportionality factor α is called the coefficient of linear expansion of the substance.

Coefficient of cubic expansion. — Let v_0 be the volume of a given substance at 0° C. and v_t its volume at t° C. The increment of volume $v_t - v_0$ is proportional to the initial volume and approximately proportional* to the rise in temperature t. That is,

$$v_t - v_0 = \beta v_0 t \qquad (120)$$

or

$$v_t = v_0(1 + \beta t) \qquad (121)$$

The proportionality factor β is called the coefficient of cubic expansion of the substance.

Remark. — The coefficient of cubic expansion is equal to three times the coefficient of linear expansion.

Proof. — Consider a cube of which the length of edge at 0° C. is l_0, at t° C. the length of edge is $l_0(1 + \alpha t)$. The volume at 0° C. is $v_0 = l_0^3$, and the volume at t° C. is $v_t = l_0^3(1 + \alpha t)^3$ or $v_t = l_0^3(1 + 3\alpha + 3\alpha^2 + \alpha^3)$, or, discarding the terms $3\alpha^2$ and α^3, and writing v_0 for l_0^3 we have:

$$v_t = v_0(1 + 3\alpha) \qquad (124)$$

Comparing this with equation (121) we find

$$3\alpha = \beta \qquad \text{Q.E.D.} \qquad (125)$$

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* Exactly proportional if the rise of temperature is small. In case the rise of temperature is great, then l_t is a function of t, such that

$$l_t = l_0(1 + \alpha t + \alpha' t^2 + \alpha'' t^3 \ldots) \qquad (122)$$

and similarly,

$$v_t = v_0(1 + \beta t + \beta' t^2 + \beta'' t^3 \ldots) \qquad (123)$$

in which α, α', α'', etc., and β, β', β'', etc. are constants for a given substance.

231. Variation of the density of a substance with temperature. — Consider a substance of mass m, of which the volume at 0° C. is v_0, and the volume at t° C. is v_t. The density of the substance at 0° C. is

$$d_0 = \frac{m}{v_0} \tag{126}$$

and the density of the substance at t° C. is

$$d_t = \frac{m}{v_t} \tag{127}$$

Dividing equation (126) by equation (127), member by member, we have

$$\frac{d_0}{d_t} = \frac{v_t}{v_0} \tag{128}$$

Substituting this value of $\frac{v_t}{v_0}$ in equation (121) we have

$$d_0 = d_t(1 + \beta t) \tag{129}$$

232. Regnault's method for determining the expansion of a liquid. — Two tubes A and B, open at top and connected by an air tube C at bottom, are filled with the liquid as shown in Fig. 125. The tube A is placed in a bath at temperature t° C., and the tube B in a bath at temperature 0° C., and the vertical distances l_0 and l_t are measured. Then:

$$\frac{d_0}{d_t} = \frac{l_t}{l_0} \tag{130}$$

whence from equation (128) we have

$$\frac{v_t}{v_0} = \frac{l_t}{l_0} \tag{131}$$

Proof of equation (130). The pressure of the air in C exceeds the outside pressure by the amount $l_0 d_0 g$, or by the amount $l_t d_t g$ from equation (83), Art. 161, so that $l_0 d_0 g = l_t d_t g$, whence equation 130 results directly.

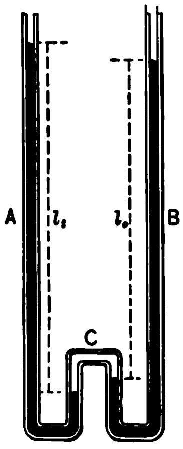

Fig. 125.

The following table gives the density and specific volume (volume of a gram of the substance) of mercury at various temperatures as determined by Regnault's method.

TABLE.

TEMPERATURE.	WEIGHT OF 1 C.M.^{3}	VOLUME OF 1 GRAM.	TEMPERATURE.	WEIGHT OF 1 C.M.^{3}	VOLUME OF 1 GRAM.
0° C.	13.5956	0.073553	190°	13.1385	0.076112
10°	.5709	.073687	200°	.1150	.076249
20°	.5463	.073821	210°	.0915	.076386
30°	.5218	.073954	220°	.0680	.076523
40°	.4974	.074088	230°	.0445	.076660
50°	.4731	.074222	240°	.0210	.076799
60°	.4488	.074356	250°	12.9976	.076937
70°	.4246	.074490	260°	.9742	.077076
80°	.4005	.074624	270°	.9508	.077215
90°	.3764	.074759	280°	.9274	.077355
100°	.3524	.074893	290°	.9041	.077495
110°	.3284	.075028	300°	.8807	.077635
120°	.3045	.075162	310°	.8573	.077776
130°	.2807	.075297	320°	.8340	.077918
140°	.2569	.075432	330°	.8107	.078060
150°	.2331	.075568	340°	.7873	.078202
160°	.2094	.075703	350°	.7640	.078345
170°	.1858	.075839	360°	.7406	.078489
180°	.1621	.075975			

233. Indirect determination of expansion of liquids. — The expansion of water and volatile liquids cannot be easily determined by Regnault's method because of the troublesome formation of vapor. The expansion of such liquids is most accurately determined as follows:

The volumes of a glass vessel at various temperatures are first determined by weighing the vessel full of mercury at those temperatures. The vessel is then used to determine the density of the liquid at the various temperatures.

The following table gives the results obtained in this way for water. Water contracts as it is heated from 0° C. to about 4° C., at which temperature its density is a maximum, and then it expands at a continually increasing rate as the temperature continues to rise. This is shown by the curve, Fig. 126, which is a graphical representation of the data given in the table.

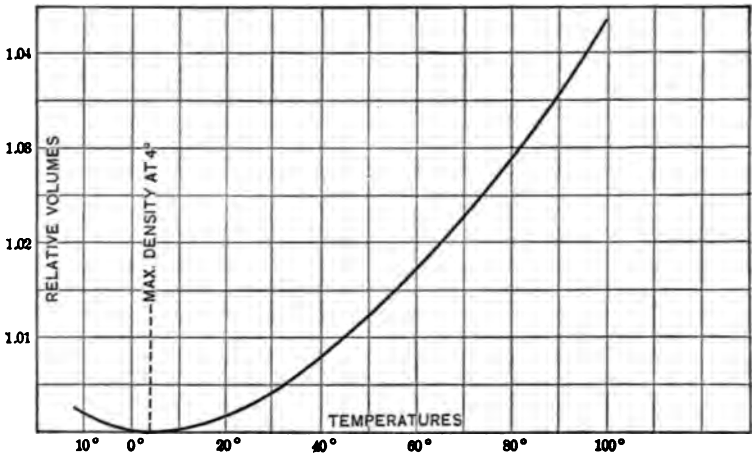

Fig. 126.

TABLE.

Densities and Specific Volumes of Water.

Temperature.	Density.	Volume.	Temperature.	Density.	Volume.
− 10° C.	0.99815	1.00186	20°	0.998252	1.001751
− 8°	0.99869	1.00131	25°	0.997098	1.002911
− 6°	0.99912	1.00088	30°	0.995705	1.004314
− 4°	0.99945	1.00055	35°	0.994098	1.005936
− 2°	0.99970	1.00031	40°	0.99233	1.00773
0°	0.999874	1.000127	45°	0.99035	1.00974
+ 1°	0.999930	1.000071	50°	0.98813	1.01201
2°	0.999970	1.000030	55°	0.98579	1.01442
3°	0.999993	1.000007	60°	0.98331	1.01697
4°	1.000000	1.000000	65°	0.98067	1.01971
5°	0.999992	1.000008	70°	0.97790	1.02260
6°	0.999970	1.000030	75°	0.97495	1.02569
7°	0.999932	1.000068	80°	0.97191	1.02890
8°	0.999881	1.000119	85°	0.96876	1.03224
9°	0.999815	1.000185	90°	0.96550	1.03574
10°	0.999736	1.000265	95°	0.96212	1.03938
15°	0.999143	1.000858	100°	0.95863	1.04315

234. Determination of the expansion of a solid. — The expansion (linear or cubic) of a solid may be determined by the direct measurement of its dimensions at various temperatures; or the expansion may be determined indirectly by measuring the volume of the solid at various temperatures by the gravimetric method, using water or mercury. See Art. 14.

CHAPTER X.

CALORIMETRY; CHANGES OF STATE.

235. Complete statement of the first law of thermodynamics. — When a given substance is heated by the dissipation of energy there is a definite relation between the amount of work done and the thermal effect produced; and *if the given substance is brought back to its initial state by being cooled by contact with another (cooler) substance, then the thermal effect produced in this other substance is the same as would be produced if it were heated directly by the dissipation of the original amount of work.* Thus a substance which is heated by the dissipation of energy receives something which is equivalent to the energy. This something is called *heat.**

Remark. — This equivalence of heat and work is somewhat restricted, as will appear in the discussion of the conversion of heat into work. (Chapter XII.)

236. Measurement of heat. — An amount of heat, for example the amount required to produce a given thermal effect, is measured when the amount of work required to produce the same effect has been determined. This measurement may be made by the direct determination of the work required to produce the given effect. The accomplishment of this direct method of heat measurement is very tedious. This is due partly to the difficulty of measuring work mechanically and partly to the difficulty of applying mechanical work wholly to the heating of a given substance. The work spent in any portion of an electric circuit, however, can be measured with considerable accuracy and it can be easily applied to the accomplishment of any given thermal effect. It is likely that this electrical method for directly measuring heat values will become the standard method.

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* A definite notion of heat can be arrived at only with the help of the molecular theory as explained in Art. 221.

Practical methods of heat-measurement. — Another method of measuring heat, which gives only indirectly the energy-values, but which may be quickly and easily carried out and is therefore adapted to general use, consists in applying the heat to be measured to the heating of water. (See Article 237.)

The melting of ice and the vaporization of water are likewise frequently employed in the measurement of heat. Methods in which these effects are employed, depend respectively upon a knowledge of the amount of heat (work) necessary to melt a gram of ice, or to convert a gram of water into steam. (See Articles 240 and 245.)

237. The heating of water by the dissipation of energy. — The relation between the amount of work dissipated in heating water and the rise of temperature produced has been repeatedly determined with great care. The results of such a determination are given in the accompanying Table, and are shown graphically in Fig. 127. The ordinates of the curve (Fig. 127) represent the amounts of work in joules required to raise the temperature of one gram of water from 0° C. to t° C. (hydrogen scale). This same quantity of work is given in the Table in the column headed E.

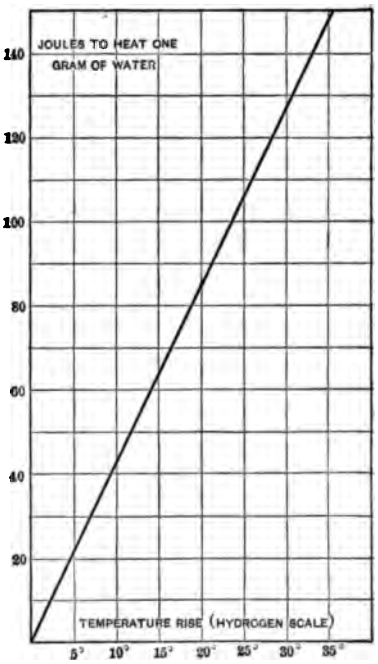

Fig. 127.

Remark. — Let E' be the work required to heat one gram of water from 0° C. to

t'° C., and E'' the amount of work required to heat one gram of water from 0° C. to t''° C. Then $E'' - E'$ is the amount of work required to heat one gram of water from t' to t''.

TABLE.*

ROWLAND'S† DETERMINATION OF THE WORK REQUIRED TO HEAT WATER.

TEMPERATURE RISE FROM 0° C. TO	ENERGY IN JOULES TO HEAT ONE GRAM OF WATER.
(t)	(E)
5°	21.040
10°	42.041
15°	63.005
20°	83.935
25°	104.834
30°	125.708
35°	146.574

Remark. — Rowland's determination was made by driving a rotatory paddle about a vertical axis at a determined speed, in a vessel of water itself mounted on a vertical axis, and prevented from turning by cords passing over fixed pulleys to weights. The torque exerted by the paddle is measured by the product of the *pull* of the cord into the lever arm thereof. This, multiplied by the angular velocity of the paddle and by the time, gives the energy expended. An accurate thermometer, projecting through the hollow axle of the paddle, indicates the rise in temperature of the water.

238. Practical units of heat. — The work required to heat a given amount of water is appoximately proportional to the rise of temperature, as is evident from the table given in Art. 237, and from the curve, Fig. 127. For many purposes this proportion is sufficiently exact and *the amount of heat required to raise the temperature of one gram of water one degree*

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* From the results reduced by W. S. Day to the hydrogen scale. (*Physical Review*, Vol. VIII., April, 1898.)

† H. A. Rowland, *Proc. American Academy of Arts and Sciences*, New Series, Vol. VII., 1878-9.

(Centigrade) may be taken as a practical* unit of heat. This unit of heat is called the *calorie*. When used in expressing the results of accurate measurements the calorie is understood to be the amount of heat required to raise the temperature of one gram of water from 14°.5 C. to 15°.5 C.

The values, in joules, of the calorie at various temperatures (hydrogen scale) are shown in Fig. 128. The standard calorie (14°.5 to 15°.5) is equivalent to 4.189 joules.

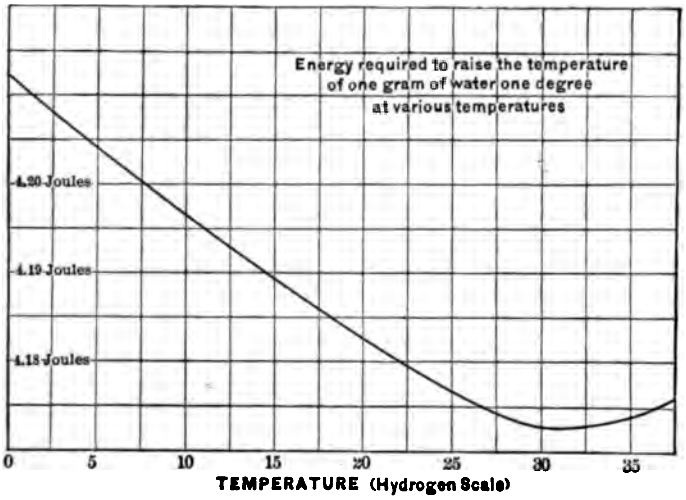

Fig. 128.

239. The water calorimeter is a vessel containing a weighed quantity of water, arranged to absorb an amount of heat to be measured. The initial temperature t' and the final temperature t'' of the water are observed. Then the energy value of the heat which has been absorbed by the water is

$$H = W(E'' - E') \qquad (132)$$

in which W is the amount of water in grams, and $E'' - E'$ is the amount of work required to heat one gram of water from t' to t''. The values of E' and E'' are given in the table in Art. 237.

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* The amount of heat required to raise the temperature of one pound of water one Fahrenheit degree is much used by engineers as a unit of heat; it is called the British thermal unit, and it is equivalent to about 778 foot-pounds.

If the heat measured by a water calorimeter is to be expressed in calories, then we have approximately

$$H_{\text{cal.}} = W(t'' - t') \qquad (133)$$

Sources of error in the use of the water calorimeter.

(*a*) A portion of the heat imparted to the calorimeter is used to raise the temperature of the containing vessel and the stirrer. The vessel and stirrer, being of the same material, are equivalent to an amount km of water, where k is the specific heat of the material and m is the combined mass of vessel and stirrer. $W + km$ is then to be used instead of W in equations (132)* and (133). In accurate work the water equivalent of the thermometer bulb must also be considered.

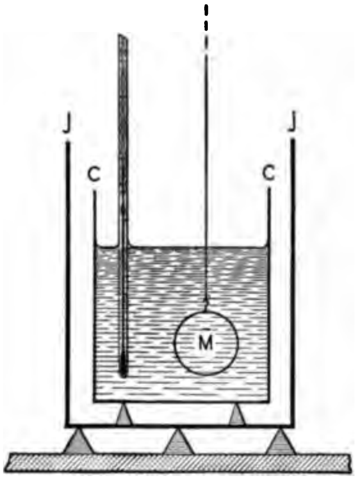

Fig. 129.

(*b*) When the calorimeter is cooler than room temperature it absorbs heat from its surroundings, and *vice versa.* This source of error is, to a great extent, obviated by so arranging that the initial temperature of the calorimeter may be as much below room temperature as the final temperature is above room temperature, and by making the duration of the experiment as short as possible. The vessel of the calorimeter *CC*, Fig. 129, is best made of thin polished metal, and it is surrounded by a polished metal jacket *JJ*, with a space between to reduce to a minimum the exchange of heat between the calorimeter and its surroundings.

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* For very accurate work a table similar to the table in Art. 237 is constructed for the material of the vessel and stirrer. Equation 132 then becomes

$$H = W(E'' - E') + m\,(e'' - e'), \qquad (134)$$

in [illegible] $e'' - e'$ is the work required to raise the temperature of one gram of the [illegible] the vessel from t' to t''.

(*c*) In a vessel of water which is being heated, there are always large local differences of temperature. In order that the indications of the thermometer may be accurate, brisk stirring is necessary.

240. The ice calorimeter. — The amount of heat (work) required to change ice at its melting point into water at the same temperature is strictly proportional to the amount of ice melted, and it has been found that 335.2 joules (80.02 calories) are required to melt one gram of ice. Heat may therefore be measured, as has been indicated in Art. 236, by determining the amount of ice it will melt. The amount of ice melted may be determined directly by weighing, or indirectly by observing the decrease in volume. The latter method is employed in the ice calorimeter of Bunsen.

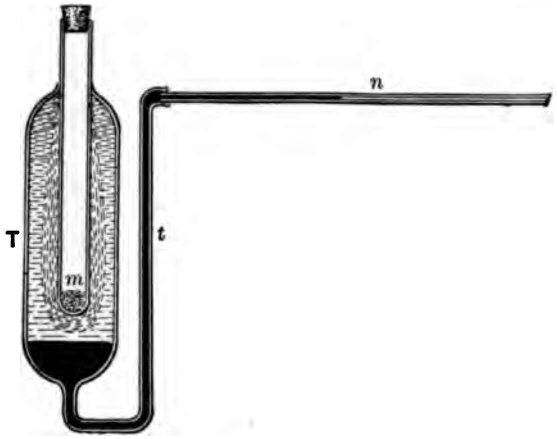

Fig. 130.

Bunsen's ice calorimeter. — This instrument, which is perhaps the most sensitive of all calorimetric devices, is shown in Fig. 130. It consists of a glass tube *T*, within which a receiving tube, in size and shape similar to an ordinary test-tube, is sealed. In the receiving tube is placed the material *m*, which gives off the heat to be measured.

The upper part of (T) is completely filled with water from which all air has been removed. The lower part is filled with mercury which fills also the narrower tube (t) and a part of the capillary neck (n). Before the determination the calorimeter is packed in a bath of melting ice. The receiving tube is then filled with a freezing mixture by the action of which it becomes encased with ice. The increase of volume drives mercury from (T) and produces a movement of the mercury column in (n). The freezing mixture having been removed from the receiving tube, the calorimeter is kept in its ice bath until the end of the mercury column comes to rest. The heated substance is then introduced into the receiving tube. The heat which it gives off in cooling to the temperature of the ice is all employed in melting the surrounding mantle of ice. The decrease in volume is indicated by the retreat of the mercury column in the neck (n), and affords a measure of the mass of ice which is melted, and so indirectly of the heat given off by the substance.

241. Specific heat of a substance. — The number of thermal units (units of work) required to raise the temperature of one gram of a given substance one degree, is called the specific heat of the substance. Specific heats are ordinarily specified in calories per gram of substance per degree rise of temperature. The specific heats of most substances (perhaps of all substances) vary with temperature.

The mean specific heat of a substance is determined by measuring the heat, H, given off by S grams of the substance in cooling from an observed temperature t' to an observed temperature t''. The mean specific heat, k, of the substance between the temperatures t' and t'' is then

$$k = \frac{H}{S(t' - t'')} \qquad (135)$$

The heat H may of course be measured by the water calorimeter or ice calorimeter. In case the water calorimeter is used, the hot substance at temperature t' is plunged into the water calorimeter at temperature t, and the final temperature t'' of the mixture is observed. The rise in temperature of the water is $t''-t$, so that from equation (133) $H=W(t''-t)$, and equation (135) becomes

$$k=\frac{W(t''-t)}{S(t'-t'')} \tag{136}$$

The accompanying tables give the mean specific heats of various substances between the temperatures indicated. The

TABLE.*

Specific Heats of Some of the Common Elements.

Substances.	Temps.	Sp. Heat.	Substances.	Temps.	Sp. Heat.
Aluminium	0°–100°	0.2185	Manganese	14°– 97°	0.1217
Antimony	0°–100°	0.495	Mercury	20°– 50°	0.0331
Bismuth	20°– 84°	0.305	Nickel	14°– 97°	0.1091
Cadmium	0°–100°	0.1804	Phosphorus:		
Carbon (diamond)	0°–100°	0.145	(Red)	15°– 98°	0.1698
" (graphite)	0°–100°	0.186	(Yellow)	13°– 36°	0.202
Copper	0°–100°	0.0933	Platinum	0°–100°	0.0323
Gold	0°–100°	0.0316	Silver	0°–100°	0.0568
Iron	0°–100°	0.1130	Sulphur	15°– 97°	0.1844
Lead	0°–100°	0.0315	Tin	0°–100°	0.0559
Magnesium	20°– 51°	0.245	Zinc	0°–100°	0.0938

Specific Heats of Compounds.

Substances.	Temps.	Sp. Heat.	Substances.	Temps.	Sp. Heat.
Bell metal	15°– 98°	0.085	Glass:		
Brass	0°	0.089	(Plate)	10°– 50°	0.186
Bronze	20°–100°	0.1043	(Crown)	10°– 50°	0.161
(88.7 Cu.+11.3 Al.)			(Flint)	10°– 50°	0.117
German silver	0°–100°	0.0946	——		
——			Paraffin	10°– 15°	0.562
Quartz	20°– 50°	0.186	Wax	26°– 42°	0.82
Granite	12°–100°	0.190	Vulcanite	20°–100°	0.331
Marble	18°– 99°	0.208	Ice	–30°– 0°	0.505

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* For more complete data, see Landolt and Börnstein, *Physikalisch-chemische Tabellen.*

ordinates of the curves, Fig. 131, show the specific heats (not mean) of iron and of carbon at various temperatures.

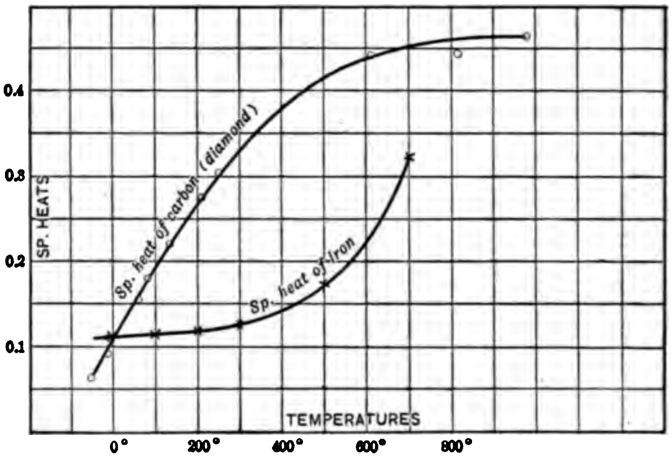

Fig. 131.

242. Heat of combustion. — Chemical action is in general accompanied by the generation or the extinction of heat. Those chemical reactions during the progress of which heat is gen-

TABLE.

HEAT OF COMBUSTION IN CALORIES PER GRAM.

SUBSTANCE.	PRODUCT OF COMBUSTION.	HEAT OF COMBUSTION.	SUBSTANCE.	PRODUCT OF COMBUSTION.	HEAT OF COMBUSTION.
Carbon (diamond) .	CO_2	7770	Illuminating gas .	—	5200 to 6365
" (graphite) .	CO_2	7762 to 8137	Raw petroleum .	—	11094
Hydrogen . . .	H_2O	34700	Refined petroleum	—	11045
Phosphorus (yellow)	P_2O_5	5747	Gunpowder . . .	—	508 to 807
Sulphur	SO_2	2220	Dynamite . . .	—	1290
Various woods . .	—	4100 to 4500	Methane (CH_4) .	—	13063
Charcoals	—	7071 to 8080	Benzol (C_6H_6) .	—	10000
Soft coals	—	7400 to 8800	Methyl alcohol .	—	5307
Anthracite . . .	—	7844	Ethyl alcohol . .	—	7183

erated are called *exothermic* reactions. Those reactions during the progress of which heat is absorbed or extinguished are called *endothermic* reactions. Combustion is the most important case of exothermic reaction. The heat generated per unit mass of a substance burned is called the heat of combustion of that substance.

The heat of combustion of a substance is usually determined by burning it in a closed metal chamber submerged in a water calorimeter. Oxygen gas from a reservoir is admitted through a tube, and the products of combustion are allowed to pass through a long metal pipe coiled in the calorimeter. The accompanying table gives the heat of combustion of a few substances in calories per gram of substance.

243. Changes of state with rise of temperature. — When heat is imparted to a solid, the temperature rises until the solid begins to melt. The temperature then remains constant until all of the substance is changed to the liquid state, when it begins to rise again and continues to rise until the liquid boils. The temperature then remains constant until the liquid is entirely changed to vapor, when it once more begins to rise. There are thus two periods during which heat is imparted to the substance without producing rise of temperature.

The *melting point* of a substance is the temperature at which the solid and liquid forms of the substance are capable of existing together in equilibrium. This temperature varies but slightly with pressure. The variation may consist either in a lowering of the melting point with pressure (case of all bodies which, like ice, expand upon freezing), or *vice versa* (case of bodies which expand upon melting).

The *boiling point* of a liquid, at a given pressure, is the temperature at which the liquid and its pure vapor can exist together in equilibrium. See Art. 244.

The following table gives the melting points of some solids at 76 cm. pressure.

TABLE.

MELTING POINTS OF VARIOUS SOLIDS.

I. *Elements.*

SUBSTANCE.	TEMPERATURE.	SUBSTANCE.	TEMPERATURE.
Aluminium	600°–850°*	Mercury	− 39°
Antimony	425°–450°	Lithium	180°
Bismuth	260°	Nickel	1476°–1517°
Bromine	70.3°	Nitrogen	− 203°–214°
Cadmium	315°	Osmium	2500° (?)
Chlorine	− 102°	Phosphorus	44°.3
Cobalt	1500°–1800°	Platinum	1757°–1780°
Copper	1050°–1100°	Potassium	62°
Gold	1030°–1250°	Selenium	217°
Iodine	113°.6	Silver	950°–1032°
Iridium	1950°–2000°	Sodium	90°
Iron	1200°–1800°	Sulphur	113°
Lead	325°	Tellurium	452°–525°
Magnesium	750°–800°	Tin	226°
Manganese	1900° (?)	Zinc	400°–420°

II. *Compounds, etc.*

Ethylen (C_2H_4)	− 169°	Methane (CH_4)	− 185
Ammonia (NH_3)	− 75°	Nitrous oxide (N_2O)	− 99°
Carbon monoxide (CO)	− 207°	Sulphur dioxide (SO_2)	− 76°
Carbon dioxide (CO_2)	− 57°	Sulphuretted hydrogen (H_2S)	− 91°
Carbon disulphide (CS_2)	− 110°		
Hydrochloric acid (HCl)	− 112°		

III. *Alloys.*

Lead 46.7%; Tin 53.3%	+ 197°	Wood's metal (Cd. 12.5%; Sn. 12.5%; Pb. 25%; Bi. 50%).	+ 70°
Britannia metal (Zn. 82%; Sb. 18%)	+ 250°		
Rose's metal (Sn. 25%; Pb. 25%; Bi. 50%)	+ 95°		

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* Where two values are given, authorities differ as to the melting point. This happens chiefly in the case of very high and very low temperatures. See Landolt and Börnstein, *Physikalische Tabellen.*

244. Minimum temperature of a vapor at a given pressure and maximum pressure of a vapor at a given temperature. — The vapor of a given substance under given pressure condenses to a liquid if an attempt is made to cool it below a certain minimum temperature. This temperature is the *boiling point* of the liquid at the given pressure.

The vapor of a given substance at given temperature condenses to a liquid if an attempt is made to increase its pressure, *e.g.* by compression, beyond a certain maximum value.

The relations between the pressure and temperature of water vapor and the vapor of ammonia respectively are given in the following tables :

TABLE.

PRESSURE AND TEMPERATURE OF SATURATED AQUEOUS VAPOR.
(FROM – 10° C. TO + 230° C.)

TEMP.	PRESSURE IN CENTIMETERS.	TEMP.	PRESSURE IN CENTIMETERS.	TEMP.	PRESSURE IN CENTIMETERS.	TEMP.	PRESSURE IN CENTIMETERS.
– 10°C.	0.2151 cm.	50°C.	9.1978 cm.	110°C.	107.537 cm.	170°C.	596.166 cm.
0°	0.4569	60°	14.8885	120°	149.128	180°	754.692
10°	0.6971	70°	23.3308	130°	203.028	190°	944.270
20°	1.7363	80°	35.4873	140°	271.763	200°	1168.896
30°	3.1510	90°	52.5468	150°	358.123	210°	1432.480
40°	5.4865	100°	76.0000	160°	465.162	230°	2092.640

TABLE.

PRESSURE AND TEMPERATURE OF SATURATED VAPOR OF AMMONIA (NH_3).
(REGNAULT.)

TEMP.	PRESSURE IN ATMOSPHERES.	TEMP.	PRESSURE IN ATMOSPHERES.
– 30° C.	1.14 atm.	20° C.	8.41 atm.
– 20°	1.83	40°	15.26
– 10°	2.82	60°	25.63
0°	4.19	80°	40.59
10°	6.02	100°	61.32

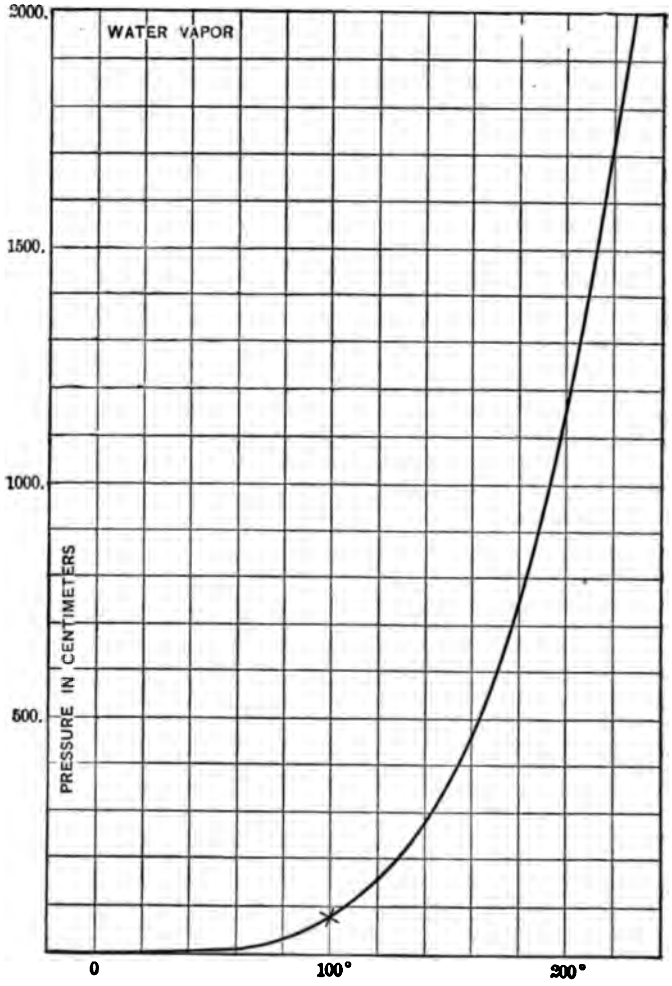
WATER VAPOR
2000.
1500.
1000.
500.
PRESSURE IN CENTIMETERS
0
100°
200°

Fig. 132.

The following are the boiling points of various liquids at 76 cm. pressure:

TABLE.

BOILING POINTS.

SUBSTANCE.	BOILING POINT.	SUBSTANCE.	BOILING POINT.
Bromine	63°.C.	Nitrous oxide (N_2O) . .	– 88°.8
Chlorine	– 33°.6	Nitric oxide (NO) . . .	– 153°.6
Mercury	357°.25	Sulphuretted hydrogen	
Nitrogen	– 193°.	(H_2S)	– 63°.5
Oxygen	– 184°.	Sulphur dioxide (SO_2) .	– 10°.
Phosphorus	285°.	Acetic acid	118°.1
Selenium	665°.	Alcohol (ethyl)	78°.40
Sulphur	448°.4	Alcohol (methyl) . . .	66°.
Zinc	929°.	Benzol	80°.36
Ethylen (C_2H_4)	– 102°.	Chloroform	51°.20
Carbon disulphide (CS_2) .	46°.04	Glycerine	290°.
Cyanogen (CN)	– 20°.7	Pentane (M)	37°.
Methane (CH_4)	– 164°.		

Saturated vapor. — A vapor is said to be saturated when it is at its maximum pressure at the given temperature, or when it is at its minimum temperature at the given pressure.

245. Variation of melting points and boiling points with pressure. — The boiling point of a liquid varies greatly with pressure as stated in Art. 244. Thus the boiling point of water is 50°C. at a pressure of 9.198 cm. of mercury, 100°C. at a pressure of 76.0 cm., 150°C. at a pressure of 358.1 cm., and 200°C. at a pressure of 1168.9 cm.; the boiling point of liquid ammonia is 30° below zero centigrade at a pressure of 1.14 atmospheres, and 60°C. at a pressure of 25.6 atmospheres.

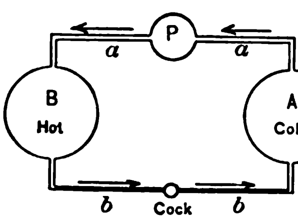

Fig. 133.

The ammonia refrigerating machine consists essentially of two chambers *A* and *B*, Fig. 133, a pump *P*, and connecting pipes *aa* and *bb*. In the chamber *A* liquid ammonia (NH_3, anhydrous) boils under low pressure, the pressure being kept low by the pump *P* which removes the vapor as fast as it is formed and compresses it into the chamber *B*, thus maintaining a high pressure in *B*.

Suppose, for example, that the pressure in *A* is kept at 1.14 atmospheres and the pressure in *B* at 25.6 atmospheres, then, according to the table given in Art. 244 the boiling point of the ammonia in *A* will be thirty degrees below zero centigrade and the boiling (or condensing) point in chamber *B* will be 60°C. Therefore chamber *A* will abstract heat from any region, a cold storage room for example, which is warmer than thirty degrees below zero; and chamber *B* will give out heat to any region, the external air for example, which is cooler than 60°C. The heat given off by chamber *B* is the heat absorbed by the chamber *A* *plus* the heat equivalent of the work done by the pump. In practice the chambers *A* and *B* are coils of pipe somewhat like the radiators used for heating buildings by steam. This type of construction facilitates the taking in of heat by the boiling liquid ammonia in *A*, and the giving off of heat by the condensing ammonia vapor in *B*. The liquid ammonia formed in *B* by the condensing vapor is allowed to flow back into *A* through a return pipe *bb* to be vaporized, pumped, and condensed over and over again.

Regelation. — The melting point of a substance varies slightly with pressure as stated in article 243. The lowering of the melting point of ice by increase of pressure is strikingly shown by the following experiment: A block of ice is supported on two pillars and a steel wire with a heavy weight on each end is hung over the block of ice. Where the wire rests against the ice the pressure is considerable and the melting point is low, say, one degree below 0° C. Some of the ice therefore immediately melts and causes * the temperature under the wire to fall to one degree below zero at which point the temperature under the wire stands so long as the pressure is maintained. The water from the melted ice flows round to the top of the wire where it is relieved of pressure and where it cannot therefore be cooler than 0° C. The result is that heat continually flows from the warm region (0° C.) on top of the wire to the cool region (— 1° C.) under the wire, the ice under the wire continues to melt, and the water as it flows round to the top of the wire continues to freeze. In the course of an hour or more a wire may thus cut its way through a large block of ice, leaving the block as one solid piece by the freezing of the water on top of the wire.

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* No solid can stand above its melting point. If one attempt to warm a solid above its melting point a portion of the solid melts and the heat which is given to the

This melting of ice at a point where it is subjected to pressure, and the immediate freezing of the resulting water when it flows out of the region of pressure is called *regelation.*

Further examples of regelation.—The remarkable ease with which a skater glides over the ice is due very largely to the formation of a thin layer of water in the region of excessive pressure under the skate runners. This water of course freezes almost instantly when the skate is past and the pressure relieved.

The cohesion of particles of ice when pressed tightly together, as exemplified in the packing of snow into balls, is due to the melting of the ice particles at the points of contact where the pressure is great and the immediate freezing of the resulting water as it flows out of the small regions of pressure. Snow must be nearly at the melting point in order that the phenomenon of regelation may be brought about by the slight pressure that can be produced in the hands.

246. Boiling and melting points of mixtures.—When a pure substance, pure water for example, has a foreign substance dis-

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substance goes to produce the change of state without causing a rise of temperature. When a block of ice at 0° C. is subjected to pressure the melting point is lowered and the ice is therefore temporarily above its melting point. The result is that a portion of the ice melts and the heat used to produce the change of state cools the ice to its melting point.

solved in it, its freezing point is lowered and its boiling point is raised. Thus when any pure metal A is alloyed with a small percentage of any other metal the melting point of the mixture is lower than the melting point of the metal A. *

When a solution freezes or boils, pure ice and pure water vapor respectively are formed, except in a few cases, and the whole of the foreign substance is left in the residual liquid. When an alloy of a metal A with a small percentage of another metal B freezes, pure crystals of metal A are, as a rule, deposited and the whole of the metal B is left in the residual liquid.

The use of ice and salt as a freezing mixture.—Ice in a strong solution of common salt has a very low melting point, fifteen or sixteen degrees below zero centigrade. Therefore ice mixed with salt falls to a temperature of fifteen or sixteen degrees below 0°C. and stands at that temperature until all the ice is melted by heat absorbed from surrounding objects. A vessel of pure water or cream surrounded by a mixture of ice and salt gives off heat to the very cold mixture until all the water or cream is frozen. Sprinkling of salt on ice or snow in the winter time does not, as commonly supposed, melt the ice; it lowers the melting point below surrounding temperatures (if these are above fifteen or sixteen degrees below 0°C.) and these surrounding objects give heat to the ice and melt it.

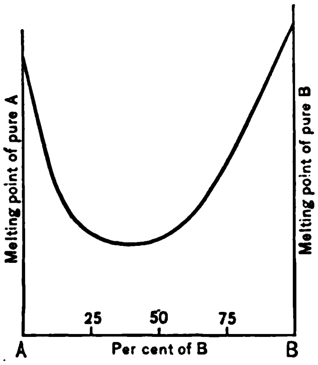

Fig. 134.

247. Melting points of alloys. Eutectic alloys.—If a *small* amount of one metal A is mixed with or dissolved in a *large* amount of metal B the resulting alloy has a lower melting point than pure metal *B*. If a *small* amount of *B* is dissolved in a *large* amount of *A* the resulting alloy has a lower melting point than pure metal *A*. The lowering of the melting point is in each case proportional to the amount of the low percentage constituent provided the percentage remains low.

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* The mathematical theory of the melting and boiling points of dilute solutions (including alloys with a small percentage of one metal) has been developed to a high degree of perfection. This theory is touched upon in appendix A.

When a larger and larger percentage of one metal *B* is mixed with another metal *A* the melting point is lowered more and more (although not according to any simple law), and the melting point reaches a *minimum value* for a certain percentage mixture of *B* and *A*; after which, increased percentage of *B* causes the melting point to rise. This is shown in Fig. 134; the ordinates of the curve representing melting points of various percentage alloys of two metals *A* and *B*.

The alloy of two metals which has the minimum melting point is called the *eutectic alloy* of those metals.

When an alloy consisting of a small amount of metal *A* and a large amount of metal *B* is melted and then slowly frozen, the pure metal *B* is at first deposited as crystals, the residual liquid grows richer and richer in metal *A*, and the freezing point is lowered more and more. This continues until the residual liquid is the eutectic alloy when further abstraction of heat causes the deposition of crystals of this alloy, the residual liquid remaining unchanged in composition, and the freezing point remaining fixed until all the residual liquid is frozen.

Castings made of an alloy of two metals consist, in general, of pure crystals of one of the metals interlaced with crystals of the eutectic alloy. Castings made from the eutectic alloy of two metals consist of crystals all of one kind, namely, crystals of the entectic alloy. Important differences of mechanical properties are associated with this difference of structure of eutectic and other alloys.

In a slowly cooled casting made from slightly impure metal, the impurity remains in solution as the outside portions of the metal crystallize so that the impurity collects in the central portion of the casting.

248. Heats of fusion and of vaporization. — When heat is applied to a substance which is melting or boiling, the temperature remains constant, and the heat-energy imparted to the substance is employed in producing the change of state.

The heat of fusion of a substance is the number of thermal units required to change unit mass of the solid at its melting point into liquid at the same temperature.

TABLE.

HEAT OF FUSION OF VARIOUS SUBSTANCES IN CALORIES PER GRAM.

Bismuth	12.640	Mercury	2.82
Bromine	16.185	Paraffin	35.10
Cadmium	13.660	Phosphorus	4.744
Glycerine	42.50	Platinum	27.18
Ice	80.02	Silver	21.07
Iron	33.50	Sulphur	9.365
Lead	5.858	Tin	13.314

The heat of vaporization of a liquid is the number of thermal units required to change unit mass of that liquid at its boiling point into vapor at the same temperature.

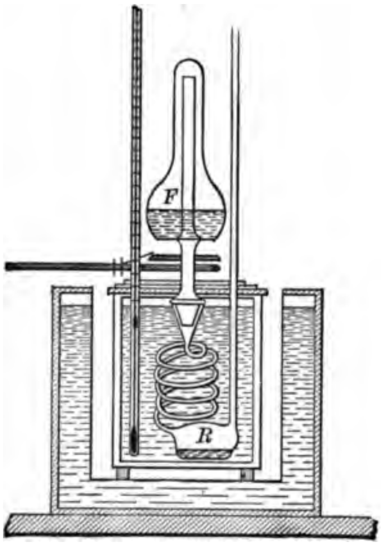

Fig. 135.

Determination of the heat of vaporization. — The apparatus employed in the measurement of the heat of vaporization is a calorimeter containing a hollow spiral and an inner receptacle. Vapor is sent into the spiral, where it is condensed, and the liquid thus produced gathers in the receiver, whence it is subsequently removed and weighed. Fig. 135 shows the form of apparatus used by Berthelot.

The liquid is heated in the vessel *F*, whence its vapor passes through the inner tube to the spiral. Here it condenses, warming the surrounding water of the calorimeter, and is collected in the receptacle *R*.

The following table gives the heat of vaporization of various liquids (at atmospheric pressure except when otherwise specified):

TABLE.

HEATS OF VAPORIZATION IN CALORIES PER GRAM.

Alcohol (ethyl)	208.92 cal.	Ether (C_4H_{10})	91.11
Ammonia (NH_3)	294.21 (at 7°.8)	Iodine	23.95
Benzol	93.45	Mercury	62.00
Bromine	45.60	Sulphur dioxide	91.7 (at 0°)
Carbon dioxide	56.25 (at 0°)	Water	535.9
Carbon disulphide	86.67		
Chloroform	58.49		

Superheating and undercooling of liquids. — When water which is free from air and dust is heated in a clean glass vessel, its temperature is likely to rise ten degrees or more above its normal boiling point, and when it begins to boil it is with almost explosive violence, the temperature falling quickly to the normal boiling point. Likewise if pure water is cooled, its temperature is likely to fall considerably below its normal freezing point. When freezing begins, a large amount of ice is suddenly formed, and the temperature quickly rises to the normal freezing point. It seems that water cannot change to vapor or to ice except there be some nucleus at which the change may begin. Most liquids show these phenomena of superheating and undercooling.

Amorphous substances, such as glass, resin, etc., have no definite melting point. These substances grow softer and softer with rise of temperature.

249. Critical temperatures. — When a liquid and its vapor — confined in a vessel — are heated, a portion of the liquid vaporizes, the pressure is increased, the density of the vapor increases, and the density of the liquid decreases.* When a certain temperature is reached, the density of the liquid and of the vapor become identical, and the vapor and liquid are physically identical. This temperature is called the *critical temperature* of the liquid.

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* At least the density of liquid and vapor become more nearly equal.

The heat of vaporization of a liquid is less, the higher the temperature (and pressure) at which the vaporization takes place, and becomes zero at the critical temperature.* For example, the heat of vaporization of water is 606.5 at 0°, 539.9 at 100°, and 464.3 at 200°.

In the following table the critical temperatures of certain substances are given:

TABLE.

CRITICAL TEMPERATURES OF VARIOUS LIQUIDS.

Alcohol (ethyl) . . .	240° C.	Methane (CH_4)	– 81° C.
Ammonia (NH_3) . .	130°	Nitrogen	– 146°
Benzol	280°	Nitrous oxide (N_2O) . . .	35°.4
Bromine	302°	Oxygen	– 118°
Carbon monoxide (CO)	141°	Sulphuretted hydrogen (H_2S)	100°
" dioxide (CO_2) .	31°	Sulphur dioxide (SO_2) . .	156°
Chlorine	141°	Turpentine oil	376°
Chloroform	260°	Water	358° (365° ?)

250. Pressures of mixed gases. — When two or more gases are mixed in a vessel, the total pressure is equal to the sum of the pressures which each component gas would exert if it occupied the vessel alone (Dalton's law). For example, if the amount of air in a vessel is such that it alone would exert a pressure p, and if the amount of water vapor in the vessel is such that it alone would exert a pressure w; then the mixture will exert a pressure $p + w$.

Therefore, a definite portion of the total pressure of a mixed gas may be considered to be due to each of the component gases of which the mixture is made. Thus the total pressure of the atmosphere is due in part to the nitrogen, in part to the oxygen, in part to the carbon dioxide, in part to the water vapor, and in part to the argon, etc., of which it is a mixture.

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* For a description of the classical experiments of Andrews upon the critical temperature, see Philosophical Transactions, 1869, II. p. 575; also Preston, Theory of Heat, p. 372.

251. Evaporation versus boiling. — It is a common observation that liquids evaporate into the air at temperatures far below their boiling points which correspond to atmospheric pressure. Thus water may evaporate at temperatures far below 100° C. A liquid at a given temperature continues to evaporate so long as the *pressure of its vapor* is less than its maximum pressure for the given temperature. This is true whether the space above the liquid is filled with vapor alone, or with vapor mixed with any gas at any pressure. For example, water vapor can exert a pressure of 35.5 cm. of mercury at 80° C. If a vessel at 80° C., containing some water, is suddenly filled with dry air at pressure p and closed, water vapor will form until the total pressure in the vessel is $35.5 + p$.

The air in an open bottle is driven out to a greater or less extent by water vapor when a small amount of water is introduced into the bottle. Thus, if the temperature is that which corresponds to half an atmosphere maximum pressure of water vapor, then half the air in the bottle will be driven out and the bottle will contain a mixture of equal volumes of air and water vapor. If the temperature is 100° C. all the air will be driven out of the bottle and the bottle will contain pure water vapor.

When a liquid actually boils the bubbles which rise through the liquid are pure vapor, and as such these bubbles must withstand the total pressure which exists in the liquid. Therefore a liquid cannot boil under a given pressure at a temperature below the minimum temperature of its vapor at the given pressure.

252. Atmospheric Moisture; Hygrometry.

Dew point. — The dew point is the temperature to which the atmosphere must be cooled in order that the water vapor present may be saturated.

Force of vapor. — That part of the pressure of the atmosphere which is due to the water vapor is called in meteorology "force of vapor." This pressure varies from nearly zero to 30 mm. or more.

Absolute humidity. — The amount of water in the air, usually expressed in grams of water per cubic meter of air, is called the absolute humidity of the air.

Relative humidity. — The amount of moisture in the air expressed in hundredths of what the air would contain were it saturated at the given temperature, is called the relative humidity. When the relative humidity is low, the air is said to be dry; when the relative humidity is high, the air is said to be moist.

The method most employed for the determination of the hygrometric elements (dew point, force of vapor, etc.) is by use of wet and dry bulb thermometers, from the readings of which the various quantities may be determined from empirical tables. Such tables are published by the U. S. Weather Bureau.

CHAPTER XI.

PROPERTIES OF GASES.

253. Restatement of laws relating to gases.

Boyle's law. — The volume of gas at constant temperature is inversely proportional to its pressure. That is

$$pv = \text{Constant} \qquad (85) \text{ bis}$$

*Gay Lussac's law.** (a) The pressure of a constant volume of gas is proportional to the absolute temperature.

(b) The volume of a given amount of a gas at constant pressure is proportional to the absolute temperature.

General formula for Boyle's and Gay Lussac's laws. — The complete relation between pressure, volume and temperature of a gas is expressed by the formula.

$$pv = MRT \qquad (137)$$

in which p is the pressure, v the volume, M the mass of the gas in the grams, T the absolute temperature, and R is a proportionality factor which has a different value for each kind of gas.

Law of integral volumes. Avogadro's principle. — Let u be the volume, reckoned at standard temperature and pressure, of one gas which combines chemically with volume v of another gas; then the ratio u/v is always a simple rational fraction ($\frac{1}{1}$ $\frac{1}{2}$ $\frac{1}{2}$ $\frac{2}{3}$ $\frac{1}{4}$ $\frac{3}{4}$ etc.) See Art. 206. This law of integral volumes, taken in conjunction with the molecular theory of chemical combination (see Art. 207) indicates that *all gases have the same number of molecules per unit volume at the same temperature and pressure.* (Avogadro.)

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* The statements here given of Gay Lussac's law involve the absolute statements of Art. 223 together with the definition of temperature ratios given in Art. 224. The statements here given are slightly misleading, inasmuch as they make it appear that absolute temperature has been previously and independently defined.

Dalton's law. — A mixture of gases, having no chemical action on each other, exerts a pressure which is the sum of the pressures which would be exerted by each component gas separately if it occupied the containing vessel alone at the given temperature. See Art. 250.

Joule and Thomson's principle, that only a small change (usually negligible) of temperature occurs when a gas is allowed to expand freely through an orifice, is discussed in a subsequent article. This law is quite as important, when we come to the discussion of the mathematical theory of gases, as the laws of Boyle and Gay Lussac.

254. The kinetic theory of gases. — Imagine a large number of very small moving particles (molecules) enclosed in a vessel. Imagine these particles to have the property of rebounding with undiminished velocity when they strike the walls of the vessel, to be so small as seldom to collide against each other, and to exert no perceptible attraction or repulsion on each other. Such a system of particles would exhibit all the properties of a gas. Therefore a gas is thought to consist of such a sytem of particles or molecules.

Let p be the pressure of a gas, v the volume of the containing vessel (which is of course the volume of the gas), N the total number of molecules in the vessel, $n\,(=N/v)$ the number of particles per unit volume, and m the mass of each molecule. Now, the kinetic energy of the system of particles is constant since the particles rebound from the walls of the vessel with unchanged velocity. Therefore the *average* kinetic energy per molecule, namely, $\frac{1}{2}m\omega^2$, is constant and definite in value; the quantity ω^2 is the average value of the square of the velocities of the various particles; and we have:

$$p = \tfrac{1}{3}nm\omega^2 \tag{138}$$

or, since $n = N/v$,

$$pv = \tfrac{1}{3}Nm\omega^2 \tag{139}$$

Proof. — The square of the velocity of a given particle is equal to the sum of the squares of the x, y and z components of its velocity. Therefore the sum of the squares of the velocities of all the particles is equal to the sum of the squares of all the x-components, *plus* the sum of the squares of all the y-components, *plus* the sum of the squares of all the z-components. The particles move at random in all directions, so that the sum of the squares of the x-components, of the y-components, and of the z-components are equal each to each. Therefore (a) *The sum of the squares,* $N\omega^2$, *of all the velocities is equal to three times the sum of the squares of the x-components.*

Imagine the containing vessel to consist of two parallel walls, of area q, distant d from each other, perpendicular to the x-axis of reference, and between which the gas is confined. Only the x-components of the molecular velocities contribute, by impact, to the pressure on these walls, so that the y and z components may be ignored. Consider a single particle, the x-component of whose velocity is a. This particle strikes first one wall and then the other, traveling back and forth $a/2d$ times per second. At each impact the velocity of the particle changes by $2a$, that is, from $+a$ to $-a$, or the momentum of the particle changes by $2am$. Therefore momentum is lost on each wall by the impact of this particle at the average rate $2am \times a/2d$, or ma^2/d, which is the average force exerted on the wall by this particle. That is, the force on one wall, due to one particle, is equal to m/d times the square of its x-velocity component. Therefore the total force F, exerted on the wall by all the particles, is equal to m/d times the sum of the squares of their x-velocity components. Therefore $F = \frac{1}{3}\omega^2 N \cdot m/d$; see ($a$). Dividing by q, and putting $qd = v$, we have $F/q = p = \frac{1}{3} Nm\omega^2/v$. Q.E.D.

The kinetic theory of gases is very important as furnishing a rational basis for Boyle's law, Gay Lussac's law, etc., and in enabling one to form clear mental pictures of the various gas phenomena as follows:

Heating of a gas. — When the walls of the containing vessel are heated the molecules of the gas rebound with increased velocity and the temperature of the gas rises. When the walls of the containing vessel are cooled the molecules of the gas rebound with diminished velocity and the temperature of the gas falls.

Heating of a gas by compression. — When a gas is compressed under a piston in a cylinder the particles of the gas rebound from the inwardly moving piston with *unchanged velocity relative to the piston* but with increased actual velocity and the temperature of the gas rises. When a gas is expanded under a receding piston in a cylinder the particles of the gas rebound from the receding piston with diminished actual velocity and the temperature of the gas falls.

Boyle's law and Gay Lussac's law. — If we assume that the **absolute temperature of a gas is proportional to the average kinetic energy per molecule,** namely, $\frac{1}{2}m\omega^2$, we may write *constant* × M × T for $\frac{1}{3}Nm\omega^2$ in equation (138) and this equation then becomes

$$pv = MRT \qquad (137) \text{ bis}$$

in which R is a constant.

Avogadro's principle is shown to be consistent with the kinetic theory of gases as follows: Consider two gases. Let p_1, n_1, m_1, and ω_1^2 be the pressure, number of particles per cubic centimeter, etc., of the one, and p_2, n_2, m_2, and ω_2^2 the corresponding quantities of the other gas. Then $p_1 = \frac{1}{3}n_1m_1\omega_1^2$, and $p_2 = \frac{1}{3}n_2m_2\omega_2^2$ from equation (138). If $p_1 = p_2$, and if the temperatures of the gases are equal, requiring $m_1\omega_1^2$ to be equal to $m_2\omega_2^2$, according to the above assumption that absolute temperature is proportional to the average kinetic energy per molecule, then $n_1 = n_2$.

Dalton's law is consistent with the kinetic theory of gases inasmuch as the moving particles are assumed to be so small that they do not interfere with each other in any way. Thus, given a number N' of oxygen molecules and a number N'' of nitrogen molecules in a containing vessel. Each set of molecules will move exactly as if the others were not present and exert the same pressure as if they occupied the vessel alone.

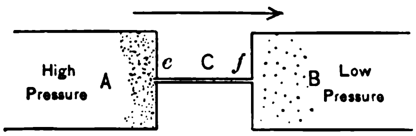

Fig. 136.

Joule's and Thomson's principle. — Gas is compressed in a vessel A and allowed to escape through a very narrow channel C into a vessel B, Fig. 136. The molecules of the gas are very close together in A, and much farther apart in B. Consider a given particle as it enters the channel C at e. Upon entering the channel this particle moves away from the great number of particles immediately behind it and if these particles attract the given particle it will lose velocity as it enters the channel. Again, the given particle will gain a little velocity as it leaves the channel at f because of its being attracted by the few particles in B, but the velocity thus gained will be less than the velocity lost at e. Therefore the gas molecules if they attract each other, will lose velocity when they pass through C, and, conversely, if the gas particles repel each other they will gain velocity when they pass through C. The losing of velocity corresponds to a cooling of the gas, and the gaining of velocity corresponds to a warming of the gas as it passes through the porous plug. See Art. 256.

255. Equation of van der Waals. — All gases deviate more or less from the laws of Boyle and Gay Lussac, and show perceptible change of temperature when allowed to expand freely. The molecules are not so small as seldom to collide, and the particles do in general attract or repel each other. When the effects of collision and of attraction (or repulsion) are taken into account in the kinetic theory a complicated form of equation (137) is arrived at as follows:

(*a*) *Effect of size of molecules.* — If the moving particles have any size, collisions and impacts take place before the centers of the particles are coincident, or before the centers of the particles are in the plane of the wall of the containing vessel. Shorter distances are thus traversed between collisions, impacts are more frequent, and the pressure is greater than it would be if the particles were indefinitely small. The result is very much as if the volume of the containing vessel were smaller by a constant amount, b, than it really is. Equation (137) may be modified so as to take account of this deviation, by writing $v - b$ for v. The value of b depends upon the amount and nature of the gas, and its value for one gram of a gas is called the *molecular volume* of the gas.

(*b*) *The effect of mutual attraction of particles* is to slow down the particles as they come into the layers of the gas adjacent to the walls. The attraction of the walls is constant and need not be considered. This slowing down of the particles makes the pressure of the gas less than it would otherwise be, by an amount which can be shown to be proportional to the square of the density of the gas or inversely proportional to the square of its volume.

Equation (137) may be modified, so as to take account of this deviation, by writing $p + a/v^2$ for p. The quantity a is a constant for a given amount of a given gas. Equation (137) therefore becomes

$$\left(p + \frac{a}{v^2}\right)(v - b) = MRT \tag{140}$$

This equation is due to van der Waals.

256. Joule and Thomson's experiment. — That the mutual attraction (or repulsion) of the molecules of a gas is small was first shown by Joule about 1848. He placed a two-chambered vessel in a water calorimeter and found no perceptible rise or fall of temperature of the calorimeter when a gas, which had been compressed in one chamber, was allowed to flow into the other chamber, which was empty, by opening a cock.

Joule and Thomson (Lord Kelvin) afterwards, in 1852, devised a method whereby this matter could be subjected to a much more searching test. The gas to be tested was forced into a pipe under high pressure and allowed to expand to low pressure by passing through a porous plug P, Fig. 137. Two thermometers T and T' served to indicate the temperature of the gas before and after expansion.

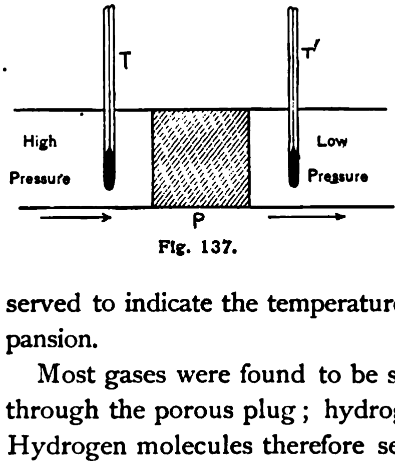

Fig. 137.

Most gases were found to be slightly cooled by the expansion through the porous plug; hydrogen alone being slightly heated. Hydrogen molecules therefore seem to repel each other at ordinary temperatures and pressures, while the molecules of air and of other gases seem to attract each other at ordinary temperatures and pressures.

The expansion of a gas through a porous plug or through an orifice is called *free expansion;* when so expanded the gas does no work, on the other hand a gas does a very considerable amount of work when it expands under a receding piston in a cylinder.

This Joule-Thomson effect, that is, the cooling of a gas by free expansion, is utilized in Linde's *liquid air machine.* This machine operates as follows: Air under very great pressure (150 to 200 atmospheres) is forced through a large coil of small copper tube at the end of which it escapes through a fine orifice into a low pressure region whence it flows back over the coil of copper pipe. The expansion of the air at the fine orifice cools the air slightly; this slightly cooled air in flowing back over the coil of copper pipe cools the inflowing air, which, in its turn, is further cooled when it passes the orifice. The inflowing air is then still further cooled in the coil of pipe, and so on until the temperature at the orifice is so greatly reduced that a portion of the air condenses into a liquid as it issues from the orifice. This liquid air collects in the low pressure chamber and is drawn off at will.

Remark 1.— The cooling of a gas by free expansion, which is so small as to be barely measureable at ordinary temperatures and pressures, is very considerable at the temperatures and pressures which prevail at the expansion orifice in a liquid air machine. Such a machine requires from fifteen minutes to an hour to get into full operation after starting.

Remark 2.— A liquid air machine so arranged that the high pressure air might expand against a moving piston would produce a greater quantity of liquid air from the same supply of compressed air, inasmuch as the very considerable cooling effect which accompanies the doing of external work by an expanding gas would contribute to the effective operation of the machine.

257. Perfect gas.— A beginner may obtain a good idea of the thermal properties of gases, sufficiently exact for all ordinary purposes, by assuming that Boyle's law, Gay Lussac's law, Joule and Thomson's principle, etc., are rigorously true. Most of the discussion given in this chapter is based on this assumption and this discussion therefore applies to an ideal gas called *the perfect gas.* All gases approach the ideal perfect gas at ordinary temperatures when highly rarefied, and the simple gases, that is, gases of which the molecule is not complex, approximate very closely to the ideal perfect gas at ordinary temperatures and pressures. All gases depart widely from the ideal perfect gas at high pressures and low temperatures.

258. Work done upon a gas during compression, and work done by a gas during expansion. — When the volume of a gas is reduced because of an increase of pressure or because of a drop in temperature, work is done on the gas by the pressure, that is to say, by the agent which maintains the pressure. The amount of work done is equal to the product of the pressure into the decrement (small) of volume. Let p be the pressure, Δv the decrement of volume, and ΔW the work done upon the gas. Then

$$\Delta W = -p \cdot \Delta v \tag{141}$$

Work *done upon* a gas is considered positive, and when Δv is a decrement it is negative and the product $-p \cdot \Delta v$ is positive as it should be. When the volume of a gas increases Δv is positive, and the product $-p \cdot \Delta v$ is negative, representing work *done by* the gas. To do work a gas must expand against a moving wall or piston. Such expansion is called *constrained* expansion to distinguish it from free expansion through an orifice.

259. Specific heats of a gas. — The number of thermal units (ergs) * required to raise the temperature of one gram of a gas one degree is called the specific heat of the gas.

If the volume of the gas does not change during the rise in temperature then, inasmuch as no external work is done by the gas, all of the heat applied goes to increase the internal energy of the gas.

If, however, the gas be allowed to expand as the temperature rises, to such extent, for example, as to keep the pressure constant, then the heat which is supplied to the gas to raise its temperature must not only increase the internal energy of the gas but also make up for the external work, $p\cdot dv$, *done by* the gas as it expands.

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* Throughout this and the succeeding chapter heat is expressed in ergs.

The specific heat of a gas has, therefore two important values namely:

C_v or the specific heat at constant volume.

C_p or the specific heat at constant pressure of which the latter has the larger value.

Relation between C_v *and* C_p. To raise the temperature of M grams of a gas dT degrees requires $C_vM\cdot dT$ units of heat (ergs) if the volume is kept constant. If then the gas is allowed to expand by an amount dv sufficient to bring the pressure back to its initial value p, an amount of work $p\cdot dv$ is *done by* the gas and an additional amount of heat equal to $p\cdot dv$ must be given to the gas to keep up its temperature, for according to Joule's and Thomson's principle the only appreciable cause of change of temperature by expansion of a gas is the loss of energy of the gas by the doing of external work. Therefore a total amount of heat $C_vM\cdot dT + p\cdot dv$ is required to raise the temperature of the gas by the amount dT when the gas is allowed to expand sufficiently to keep its pressure constant; but according to the definition of C_p, the amount of heat required for this change of temperature with constant pressure is $C_pM\cdot dT$. Therefore

$$C_pM\cdot dT = C_vM\cdot dT + p\cdot dv \qquad (142)$$

260. Proposition. — The constant R, equation (137), for a given gas is equal to the difference between the specific heat of the gas at constant pressure (C_p) and the specific heat of the gas at constant volume (C_v). That is

$$C_p - C_v = R \qquad (143)$$

Proof. — Imagine M grams of the gas to have been increased in temperature by the amount dT at constant pressure; dv being the increment of volume. Then according to equation (142) we have:

$$C_pM\cdot dT = C_vM\cdot dT + p\cdot dv \qquad \text{(i)}$$

U of M

The increment of volume necessary to keep the pressure constant when the temperature increases is found by differentiating equation (137) remembering that T and v, only, vary. This gives

$$p \cdot dv = MR \cdot dT \qquad \text{(ii)}$$

Substituting this value of $p \cdot dv$ in equation (i) we have equation (143).

261. Isothermic expansion and compression. — When the temperature of a gas is kept constant by supplying heat to the gas when the gas expands, or by abstracting heat from the gas when the gas is compressed, the expansion or compression is said to be *isothermic*. Equation (137) gives the relation between p and v of a perfect gas during isothermic expansion or compression, T being a constant as well as M and R.

The full line curves in Fig. 138 show graphically the isothermic relation between p and v for several different temperatures. The ordinates of these curves represent pressures and the abscissas represent volumes. The curves are sometimes called the *isothermal lines* of the gas; they are equilateral hyperbolas.

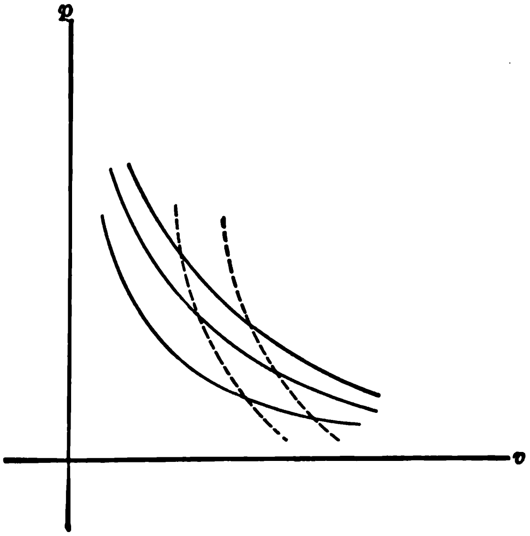

Fig. 138.

262. Adiabatic or isentropic expansion and compression.—When a gas is expanded or compressed and not allowed to receive heat from, or to give heat to surrounding bodies, the expansion or compression is called *adiabatic* or *isentropic.*[*] The relation between p and v of a perfect gas during isentropic expansion or compression is:

$$pv^k = \text{constant} \tag{144}$$

in which

$$k = \frac{C_p}{C_v} \tag{145}$$

The dotted curves in Fig. 138 show graphically the isentropic relation between p and v (for two different values of the entropy).

Remark.—The value of the ratio k for air is 1.41.

Proof of equation (144). — Consider a small isentropic expansion and let dv, dp and dT be the actual changes of volume, pressure and temperature; dv being an increment, while dp and dT are both decrements and therefore both negative.

The work $p \cdot dv$ *done by* the gas is all made up by the decrement $MC_v \cdot dT$ of the internal energy of the gas; therefore, remembering that $MC_v \cdot dT$ is negative while $p \cdot dv$ is positive we have:

$$p \cdot dv = -MC_v \cdot dT \tag{i}$$

From the equation $pv = MRT$ (137) we have:

$$T = \frac{1}{MR} \cdot pv \tag{ii}$$

whence

$$dT = \frac{1}{MR}(p \cdot dv + v \cdot dp) \tag{iii}$$

Substituting this value of dT in equation (i), remembering that $R = C_p - C_v$ (143), and that $\kappa = C_p/C_v$ (145), we have:

$$\frac{dp}{p} + \kappa \frac{dv}{v} = 0 \tag{iv}$$

whence by integrating

$$\text{Log}\, p + \kappa \,\text{Log}\, v = \text{constant}$$

or

---

[*] The term *isentropic* expansion is preferable to the term *adiabatic* inasmuch as a certain quantity called the entropy of the gas is constant during this kind of expansion.

$$\text{Log}\,(pv^{\kappa}) = \text{constant}$$

or

$$pv^{\kappa} = \text{constant}$$

Q.E.D.

263. Rise in temperature of a gas during isentropic compression. —We may eliminate p (or v) from equation (144) by substituting the value of p (or v) from equation (137). In this way we find:

$$Tv^{\kappa-1} = \text{constant} \tag{146}$$

or

$$Tp^{\frac{1-\kappa}{\kappa}} = \text{constant} \tag{147}$$

during isentropic expansion or compression.

Examples of isentropic expansion and compression. — The expansion and compression of the air in sound waves is isentropic, inasmuch as the expansion and compression take place so quickly that the expanded or compressed portion of the air does not have time to give off or to receive any appreciable amount of heat.

When a large mass of air moves upwards, as, for example, when a wind blows up a mountain slope, the pressure of the mass of air falls off with increasing altitude and the temperature is reduced. The expansion is isentropic, inasmuch as the mass of air is so large that it cannot, during the brief time of the ascent, receive or give off an appreciable amount of heat. Suppose that the rising mass of air was initially at temperature T_1 and pressure p_1, and that its pressure has fallen to p_2. The temperature T_2 corresponding to p_2 may be found from equation (147), which gives

$$T_1p_1^{\frac{1-\kappa}{\kappa}} = T_2p_2^{\frac{1-\kappa}{\kappa}} \tag{148}$$

A very interesting phenomenon due to the cooling of a rising column of air by isentropic expansion is the formation of the beautiful cumulus clouds on a quiet summer day. The warm moist air near the ground starts streaming upwards through the superimposed cold air. This upward stream once started draws like a chimney and the rising column develops until it becomes very large and very high. At a very sharply defined altitude the pressure reaches a value for which the temperature [according to equation (148)] of the rising air is reduced to the dew point. The further cooling which is produced as the air passes above this level, causes the condensation of water vapor and the formation of mist or cloud. The strikingly flat bottoms of these cumulus clouds are at that altitude where the rising air reaches the temperature of the dew point.

The condensation of water vapor as rain is due very largely to the isentropic cooling of the great rising column of air, sometimes hundreds of miles in diameter, at the center of what is technically called a cyclone.

When a gas is quickly compressed under a piston in a cylinder, the compression is isentropic. If the initial volume is v_1, initial temperature T_1, and final volume v_2, we may find the temperature T_2 corresponding to v_2 from equation (146), which gives

$$T_1 v_1^{\kappa-1} = T_2 v_2^{\kappa-1} \tag{149}$$

264. Method of Clement and Desormes for the experimental determination of κ. — A vessel filled with the gas at pressure p_1 and temperature T_1 is opened to the air, allowing the pressure to drop quickly to atmospheric pressure p_2, the temperature falling at the same time to T_2. Equation (148) gives

$$T_1 p_1^{\frac{1-\kappa}{\kappa}} = T_2 p_2^{\frac{1-\kappa}{\kappa}} \qquad \text{(148) bis}$$

The vessel is then quickly closed and allowed to stand until it has reached its former temperature T_1, when the pressure is p_3, so that from Gay Lussac's law we have

$$\frac{T_2}{p_2} + \frac{T_1}{p_3} \qquad \text{(i)}$$

Substituting the value of T_2 from (i) in (148) and reducing, we have

$$\left(\frac{p_1}{p_2}\right)^{\frac{1-\kappa}{\kappa}} = \frac{p_2}{p_3} \qquad \text{(ii)}$$

from which, p_1, p_2, and p_3 having been observed, κ may be calculated. The value of κ for air is 1.41.

Remark. — The numerical value of the constant R equation (137) is easily calculated from the observed value of the density M/v of the gas at a known temperature T and pressure p. Therefore the value of $(C_p - C_v)$ is known for the gas from equation (143). The difference $(C_p - C_v)$ being thus known and the ratio C_p/C_v being determined by Clement and Desormes' method, one may easily calculate the values C_p and C_v.

265. The two bulk moduli of a gas.* Consider a decrement of volume dv produced by an increment of pressure dp. Then dv/v is the isotropic strain produced by the stress dp, and the quotient *stress divided by strain or dp divided by dv/v is the bulk modulus* of the gas. That is

$$M = v \cdot \frac{dp}{dv} \tag{150}$$

in which M is the bulk modulus of a gas. Now the decrement of volume produced by the increment of pressure Δp has one value if the gas is allowed to give off heat so that its temperature may not change and a distinctly different and smaller value if the gas is not allowed to give off heat. Therefore, the bulk modulus of a gas has two important values, namely :

M_t, the isothermic modulus, corresponding to compression at constant temperature.

M_e, the isentropic modulus, corresponding to adiabatic or isentropic compression.

The isothermic bulk modulus has the smaller value.

Value of isothermic modulus. — From equation (137) we may find the relation between dp and dv for isothermic compression. Differentiating (137) we have, T being constant :

$$p \cdot dv + v \cdot dp = 0 \tag{i}$$

whence, using equation (150), we have :

$$v \cdot \frac{dp}{dv} = M_t = -p \tag{ii}$$

The negative sign is fictitious and comes in because of the fact that in the definition of bulk modulus dv is positive when it is a decrement while in equation (i) dv is positive when it is an increment. Therefore we may write:

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* The reader must not confuse the M in this article with the M in equation (137).

$$M_t = p \tag{151}$$

in which M_t is the isothermic bulk modulus of a gas.

Value of isentropic modulus. — From equation (144) we may find the relation between dp and dv for isentropic compression. Differentiating (144) we have

$$\kappa p v^{\kappa-1} \cdot dv + v^{\kappa} \cdot dp = 0 \tag{iii}$$

whence, using equation (150), we have:

$$v \cdot \frac{dp}{dv} = M_e = -\kappa p \tag{iv}$$

The negative sign is fictitious here as in equation (ii). Therefore we may write:

$$M_e = \kappa p \tag{152}$$

in which M_e is the isentropic bulk modulus of a gas.

266. The velocity of sound in a gas. — The velocity of sound in a gas is equal to the square root of the quotient of the bulk modulus of the gas divided by the density; and of course, since the expansion and compression which occur in sound waves are isentropic, it is the isentropic bulk modulus which enters into this formula. That is

$$V = \sqrt{\frac{M_e}{\rho}} \tag*{(i)*}$$

in which V is the velocity of sound in a gas, ρ is the density of the gas, and M_e is the isentropic bulk modulus of the gas. Substituting the value of M_e from equation (152) in equation (i) above, we have:

$$V = \sqrt{\frac{\kappa p}{\rho}} \tag{ii}$$

from which the ratio, κ, of the specific heats of the gas may be calculated when the velocity of sound in the gas has been observed at a given temperature and the ratio p/ρ has been determined for the same temperature.

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* A derivation of this equation is given in appendix A of Vol. III. of Nichols and Franklin's "Elements of Physics."

CHAPTER XII.

THERMODYNAMICS.

267. Specification of physical state. — The physical state of a *given amount* of any fluid * is completely fixed or specified, when its *pressure* and *volume* are given. Throughout this chapter *volume* will be understood to mean the volume of a gram of the fluid under consideration.

The physical state of a fluid is also fixed when its pressure and temperature are given or when its temperature and volume (of a gram) are given. Specification of state in terms of pressure and volume is to be preferred.

Example. — A gram of water, to occupy five cubic centimeters of volume at a pressure of 50 cm. of mercury † must be at a certain temperature and a certain portion of the water must be liquid and a certain portion vapor.

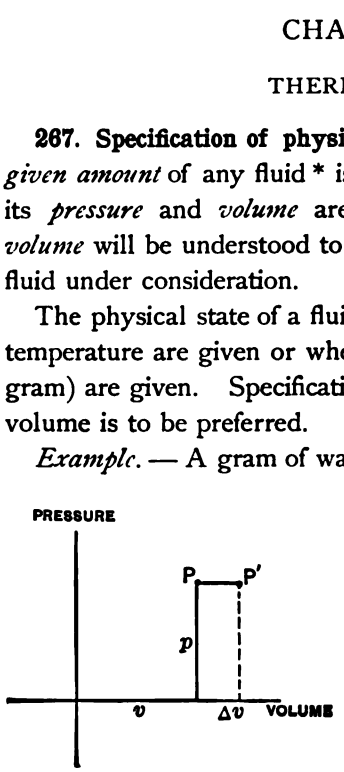

Fig. 139.

268. Watt's diagram. — It is convenient to represent the physical state of a fluid by a point *P*, Fig. 139, of which the abscissa represents the volume and the ordinate the pressure of the fluid.

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* The thermodynamics of solids is very greatly complicated because of the possibility of other kinds of stress than hydrostatic pressure. The object of this chapter is to bring out some of the most important facts of thermodynamics in the simplest possible way and therefore the details of the discussion will be limited chiefly to fluids. Where a statement is generally true the word *substance* or *system* (of substances) will be used instead of the word *fluid*.

† Pressures in this and in the preceding chapter are understood to be expressed in dynes per square centimeter unless otherwise distinctly specified.

When a fluid expands it does work if allowed to push against a piston, for example; and the work, $p\Delta v$, done by the fluid during an increment in volume Δv is represented by the area $p\Delta v$ as shown in Fig. 139.

269. Process. Work done on a fluid. A substance which passes slowly from one physical state to another, because of changing pressure, or volume, or both, is said to undergo or perform a *process*. The points in Watt's diagram which represent the successive states passed through by a fluid during a process form a continuous * curve called the *process curve*. The changing state of a fluid while undergoing a process is represented by a moving point Q, Fig. 140, which travels along the process curve as the fluid undergoes the process. As the point Q moves along the process curve PP' the ordinate to the point Q sweeps over the area $aPP'b$ and this area represents the total work done by the fluid or done upon the fluid during the process. It is customary in thermodynamics to consider work done upon a fluid as positive, and work is done upon a fluid during decrease of volume. Therefore the area under a process curve represents positive work when this area is swept over to the left by the ordinate to the moving point Q, and it represents negative work when it is swept over to the right by the ordinate to the moving point Q.

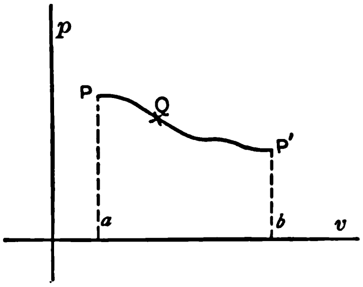

Fig. 140.

Examples. (*a*) A gas expanded or compressed in a cylinder at constant temperature is an example of a process, and in this case the process curve is an equilateral hyperbola as explained in article 261 and as shown by the full line curves in Fig. 138.

(*b*) A gas expanded or compressed in a cylinder and not allowed to give heat to or receive heat from the cylinder is an example of a process, and in this case the process curve is a variety of hyperbola as explained in article 262, and as shown by the dotted curves in Fig. 138.

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* Reversible processes, only, can be represented in Watt's diagram.

(*c*) Water vapor slowly compressed in a cylinder at constant temperature is an example of a process. In this case the process curve is as shown in Fig. 141. The portion *A* of the curve corresponds to that stage of the process during which the water vapor is below the maximum pressure for the given temperature; during this stage the pressure rises as the volume decreases and this rise of pressure continues until the maximum pressure is reached. The portion *B* of the curve corresponds to that stage of the process during which the vapor is being condensed to liquid as the volume decreases; during this stage the pressure is constant. The portion *C* of the curve corresponds to that stage of the process when the vapor has been wholly converted to liquid; during this stage the pressure must increase to enormous values to produce perceptible decrease in volume.

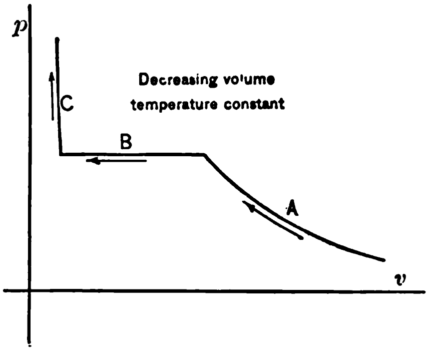

Fig. 141.

270. Cyclic process. A process is said to be *cyclic* when the substance comes back to its initial state at the end of the process. The closed curve in Fig. 142 represents a cyclic process; the fluid starting from the state *P* passes through the states represented by the successive points on the closed curve and returns to the initial volume and pressure.

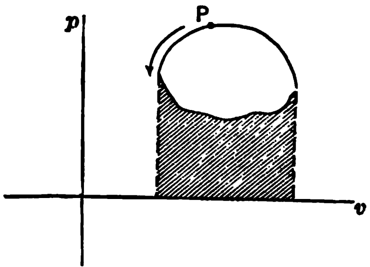

Fig. 142.

Imagine a point Q to travel around the closed curve in the direction of the arrow, as shown in Fig. 142. Then the ordinate to Q will sweep over the shaded area Fig. 142 once to the right and once to the left, while the enclosed area will be swept over once only and to the left. Therefore the *enclosed area* represents the *net* work done upon the fluid during the cyclic process. If the process is performed in the reverse direction then the enclosed area is swept over once only and to the right and it represents the *net* work done by the fluid during the cyclic process.

271. Thermal equilibrium. When a substance is shielded from outside disturbance it settles to a state in which there is no tendency to further change of any kind. Such a state is called a state of thermal equilibrium as pointed out in article 219.

The molecular conception of thermal equilibrium. — The molecular motion at a given point in a gas (and no doubt in any substance) in thermal equilibrium is entirely erratic; an irregular and extremely rapid succession of fits and starts occurs as the molecules collide against each other, and the character of the molecular motion at the point is still further complicated by the fact that different molecules are continually passing the given point from every direction and with every variety of speed and oscillatory motion. Because of the enormous number of molecules in any perceptible volume of a substance it is the *average character* of the molecular motion, only, which has to do with temperature and pressure and in general with all thermal properties of substances; and, because of the enormous number of molecules, this average character of molecular motion is constant and uniform throughout a substance when the substance is in thermal equilibrium.

272. Reversible processes. — A substance in thermal equilibrium is generally under the influence of external agencies. Thus, surrounding substances confine the given substance to a certain region of space and they exert upon the given substance a constant pressure; surrounding substances are at the same temperature as the given substance and the molecules of the given substance rebound from surrounding substances with their motion, on the average, unchanged; surrounding substances may exert constant magnetic or electric influences upon the given substance, and so on. However, a substance can not be in thermal equilibrium when work is continually done upon or done by it, or when heat is continually given to or taken from it.

If the external influences which act upon a fluid in thermal equilibrium are made to change *very slowly*, causing the pressure and volume of the fluid to pass very slowly through a continuous series of values and in general involving the doing of work upon or by the fluid and the giving of heat to or taking of heat from the fluid, the fluid will pass slowly through *a process consisting of a continuous series of states of thermal equilibrium.* Such a process is called a *reversible process*, for the reason that the fluid will pass through the same series of states in reverse order if the external influences are changed slowly so as to make the pressure and volume of the fluid pass through the same series of values in reverse order.

The characteristics of a reversible process are therefore as follows:

(*a*) A substance which undergoes a reversible process must be under varying *external* influence. A closed system can not perform a reversible process.

(*b*) A substance as it undergoes a reversible process is at each instant in a state of thermal equilibrium; and if, at a given instant during a reversible process, the external influences should cease to change, causing a sudden cessation of the doing of work on or by the substance, and of the interchange of heat between the substance and its surroundings, no commotion would be left in the substance.

(*c*) A reversible process must take place slowly, strictly with infinite slowness. An actual process, that is, a process which actually does proceed, can be only approximately reversible. Examples of reversible processes are given in the article on trailing sweeps, for it is important that it be clearly recognized that a reversible process is the limit which a trailing sweep approaches when it is performed more and more slowly.

273. Sweeping or irreversible processes. — While a substance is settling or tending to settle to thermal equilibrium it may be said to undergo a process. Such a process cannot, in general, be arrested and maintained at any stage short of complete thermal equilibrium, but always and inevitably proceeds towards that state. Such a process is, therefore, called a *sweeping process* or simply a *sweep*.

A sweeping process takes place in one direction only, that is, if A and B are two successive stages of a sweep, stage B following stage A, then stage B grows out of stage A inevitably, but stage A cannot be made to follow or to grow out of stage B by any means whatever. A sweeping process, therefore, is irreversible.

Molecular conception of the sweeping process. — While a gas (and perhaps any substance) is settling to thermal equilibrium, immediately after an explosion, for example, the character of the molecular motion at a given point changes rapidly from instant to instant and the character of the molecular motion at a given instant varies greatly from point to point in the gas; in other words, the gas is the seat of more or less violent turbulence while it is settling to thermal equilibrium.

The mutual collision among the molecules of a gas, the collision of the molecules against the walls of the containing vessel and the confused movement * of the gas molecules from one part of the vessel to another part, all tend to even up the differences in the character of molecular motion in different parts of the vessel. On the other hand, the external influences which can be brought to bear on a substance act on all the molecules in the same general way, so that the tendency of a turbulent state of a gas to die away on account of the internal actions just pointed out cannot be counteracted by external influences,* and, therefore, a sweeping process cannot be arrested or reversed by any means.

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* If a great number of white and black balls are placed in a box and shaken up, the confused motion tends to cause an even distribution of white and black balls throughout the box, for the reason that, of all possible arrangements of the balls, approximately even distribution is the most probable.

Remark. — A fluid not in thermal equilibrium has no definite pressure or temperature, and the volume of a turbulent gas pertains strictly to the containing vessel only.

274. Simple sweeps. — The settling of a closed system to thermal equilibrium is called a *simple sweep.*

Examples. — The equilibrium of a mixture of oxygen and hydrogen in a closed vessel may be disturbed by a minute spark, and the explosion and subsequent settling of the aqueous vapor to a quiescent state without loss of heat constitute a simple sweep. The equilibrium of a gas confined under high pressure in one half of a two-chambered vessel may be disturbed by opening a cock which connects the two chambers, and the rush of gas into the empty chamber constitutes a simple sweep.

275. Trailing sweeps. — When external influences change continuously, a substance in its tendency to settle to equilibrium never catches up, as it were, with the changing conditions, but trails along behind them, and we have what is called a *trailing sweep.*

Examples. — The rapid expansion † or compression of a gas in a cylinder is a trailing sweep. So long as the piston moves at a perceptible speed, the gas, in its tendency to settle to equilibrium, never catches up with the varying conditions. This is evident, for any one can see that a sudden stoppage of the piston would leave some slight turbulence in the gas, which would not be the case if the gas were in equilibrium at the instant the piston is stopped. When the piston is moved more and more slowly, the departure of the gas from strict thermal equilibrium at each stage of the expansion or compression becomes less and less, and the expansion or compression approaches more and more nearly to a reversible process.

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* The maintenance of an unending state of turbulence in a trailing sweep because of rapidly changing external influence is by no means a case in which the tendency of a turbulent state to die away is counteracted by external action, but rather a case in which the goal, namely the final state of thermal equilibrium, is made to recede continuously.

† The different effects here mentioned in connection with rapid versus slow expansion must not be confused with the distinction between isothermic and isentropic expansion of gases (see articles 261 and 262). Isothermic expansion or compression must be slow in order that the necessary heat may be given to or taken from the gas to keep its temperature constant. On the other hand, any approximation to isentropic expansion or compression of a gas in a cylinder must be rapid in order that there may be no appreciable amount of heat given to or taken from the gas by the cylinder walls.

The rapid heating (or cooling) of a gas in a closed vessel is a trailing sweep. So long as heat is given to the gas at a perceptible rate there will be perceptible differences of temperature in different parts of the gas; the gas in its tendency to settle to thermal equilibrium never catches up with the increasing temperature of the walls of the containing vessel.

When the gas is heated (or cooled) more and more slowly, that is, when heat is given to the gas at a rate which becomes more and more nearly imperceptible, then the departure of the gas from strict thermal equilibrium at each stage of the heating process becomes less and less, and the heating (or cooling) approaches more and more nearly to a reversible process.

276. Steady sweeps. — A substance may be subjected to external action which, although unvarying, is incompatible with thermal equilibrium. When such is the case the substance settles to a permanent or unvarying state which is not a state of thermal equilibrium. Such a state of a substance is called a *steady sweep.*

Examples. — The two faces of a slab or the two ends of a wire may be kept permanently at different temperatures. When this is done the slab or wire settles to an unvarying state which is by no means a state of thermal equilibrium. Heat flows through the slab or along the wire from the region of high temperature to the region of low temperature, *never* from the region of low temperature to the region of high temperature. This flow of heat through the slab or along the wire is an irreversible process and it constitutes a steady sweep.

The ends of a wire may be kept permanently at different electric pressures, for example, by connecting the wire to the terminals of a battery or dynamo. When this is done a steady electric current flows along the wire, the battery does work steadily on the wire, and this work reappears steadily as heat in the wire. Reversal of the current *does not* reverse this process and cause heat energy to disappear in the wire (cooling the wire) and reappear as work done on the battery by the wire, but the process is irreversible and it constitutes a steady sweep.

277. Thermodynamic degeneration and regeneration.—A sweeping process always plays a certain havoc, or effects a certain *degeneration* in a system. Thus, there is a certain degeneration associated with the escape of a compressed gas through an orifice; there is a certain degeneration associated with the flow of heat from a region of high temperature to a region of low temperature; there is a certain degeneration associated with the direct conversion of work into heat, and so on.

In a simple sweep the degeneration lies wholly in the relation between the initial and final states of the substance which undergoes the sweep, inasmuch as, in a simple sweep, no outside substance is affected in any way, no work is done on or by the substance which undergoes the sweep, and no heat is given to or taken from it.

In a trailing sweep the degeneration may lie partly in the relation between the initial and final states of the substance which undergoes the sweep, partly in the direct conversion of work into heat and partly in the direct transfer of heat from regions of high temperature to regions of low temperature.

In a steady sweep the substance which undergoes the sweep remains entirely unchanged as the sweep proceeds, and the degeneration lies wholly in the direct conversion of work into heat, in the direct transfer of heat from a region of high temperature to a region of low temperature, or both.

A substance which has undergone a sweeping process may be brought back to its initial state, or *regenerated*, by a reversible process; but when a substance is regenerated by a reversible process, the external action necessary to bring about the reversible process involves an equal degeneration of some external substance; that is, the regeneration of a substance by a reversible process always involves the creation of an equal external degeneration. This is, in fact, a statement of the second law of thermodynamics.

The entire subject of thermodynamics, in so far as it does not have to do with the specific thermal properties of particular substances, is based upon the consideration of the two kinds of thermodynamic degeneration which are involved in steady sweeps, that is, upon: (*a*) The thermodynamic degeneration which is represented by the direct conversion of work into heat, and the thermodynamic regeneration which is represented by the conversion of heat into work by a reversible process; and (*b*) The thermodynamic degeneration which is represented by the direct transfer of heat from a region of high temperature to a region of low temperature, and the thermodynamic regeneration which is represented by the transfer of heat from a low temperature region to a high temperature region by a reversible process.

The following two propositions concerning the two kinds of thermodynamic degeneration (*a*) and (*b*) follow at once from a consideration of steady sweeps.

PROPOSITION (A). — The thermodynamic degeneration represented by the direct conversion of work into heat at a given temperature is *proportional* to the quantity of work so converted.

Proof. — Consider a steady flow of electric current * in a wire.

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* If the reader is not familiar with the phenomena of the electric current, this discussion of the thermodynamic degeneration represented by the direct conversion of work into heat may be based upon a consideration of the conversion of work into heat

This process being steady, the amount of degeneration occurring in a given interval of time must be proportional to the time. The amount of work converted into heat is also proportional to the time. Therefore the amount of degeneration is proportional to the amount of work converted into heat. Of course the temperature must be invariable, or the process cannot be thought of as remaining identically the same from instant to instant. The dependence of this kind of degeneration upon temperature will be considered later.

Corollary. — The thermodynamic regeneration which is represented by the conversion of heat at a given temperature into work by a reversible process is proportional to the heat so converted.

PROPOSITION (B). — The thermodynamic degeneration represented by the direct transfer of heat from a given high temperature T_1 to a given low temperature T_2 is proportional to the quantity of heat transferred.

Proof. — Consider a steady flow of heat from temperature T_1 to temperature T_2 constituting a steady sweep. This process being steady, the degeneration occurring in a given interval of time must be proportional to the time. The quantity of heat transferred is also proportional to the time. Therefore, the amount of degeneration is proportional to the quantity of heat transferred. The dependence of this kind of degeneration upon temperature will be considered later.

Corollary. — The thermodynamic regeneration which is represented by the transfer of heat from a low temperature T_2 to a high temperature T_1 by a reversible process is proportional to the heat transferred.

278. The second law of thermodynamics.

(*a*) *The degeneration of a system which accompanies a sweeping*

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in the bearing of a steadily rotating shaft, or of the conversion of work into heat in a pipe through which a steady stream of water flows; although neither of these cases is an example of a strictly steady sweep inasmuch as some physical changes always proceed in a bearing, and perceptible degrees of turbulence always exist in a bearing and in a stream of water.

process cannot be directly repaired, nor can it be repaired by any means without the creation of a compensating degeneration in some other system.

This is an entirely general statement of the second law. The *direct repair* of the degeneration due to a sweeping process means the undoing of the havoc wrought by the sweep by allowing the sweep to *perform itself backwards!* This notion of *direct repair* is introduced into this general statement of the second law in order that each of the following particular statements of the law, namely, (*b*), (*c*), and (*d*), may correspond exactly in form to the general statement (*a*). A slightly modified general statement of the second law is the following:

(*a'*) *Thermodynamic degeneration and regeneration are always balanced in a reversible process, while degeneration always exceeds regeneration in any process which is in any way sweeping in character.*

(*b*) *Heat cannot pass directly from a cold body to a hot body, nor can heat be transferred from a cold body to a hot body by any means without compensation.*

(*c*) *Heat cannot be converted directly into work, nor can heat be converted into work by any means without compensation.*

The direct conversion of heat into work [see discussion following (*a*) above] would be simply the reverse of any of the ordinary sweeping processes which involve the degeneration of work into heat. Thus, work is degenerated into heat in the bearing of a rotating shaft, and we all know that to reverse the motion of the shaft will not cause the bearing to grow cold and the heat so lost to appear as work helping to turn the shaft!

(*d*) *A gas cannot pass directly* from a region of low pressure*

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*The repeated statement of self-evident facts in these statements of the second law of thermodynamics may seem ridiculous to the intelligent reader, but it must be remembered that but few persons realize that the second law of thermodynamics is a statement of a fact which every one knows, together with a generalizing clause which when once thoroughly understood is almost if not quite self-evident. Here is one more statement of the second law, the oldest English version of it:

to a region of high pressure, nor can a gas be transferred from a region of low pressure to a region of high pressure by any means without compensation.

The compensation involved in the transfer of a gas from a region of low pressure to a region of high pressure by means of a pump is the degeneration into heat of the work spent in driving the pump; a gas is always heated by compression and the work spent in driving an ideal frictionless pump would all be converted into heat.

279. Engines and refrigerating machines. — The further development of the subject of thermodynamics depends upon the establishment of the exact relation between the degeneration which is represented by the transfer of heat from a high to a low temperature and the degeneration which is represented by the conversion of work into heat. To establish this relation it is necessary to consider a *reversible process* in which the degeneration of heat from high to low temperature is compensated by the regeneration of heat into work, or *vice versa.*

The engine is a machine which determines such a process. The ordinary engine, indeed, is subject to friction, and the steam as it passes through the engine does not undergo a reversible process; but if the engine were frictionless, if it were driven slowly, if the cylinder were prevented from cooling the steam, and if the steam were expanded sufficiently to prevent puffing, then the processes involved in the operation of the engine would be reversible. Such an ideal engine we will call a *reversible engine* or a *perfect engine.*

The engine takes heat from the boiler, converts a certain portion of this heat into work, and delivers the remainder to the con-

Humpty Dumpty sat on a wall.
Humpty Dumpty had a great fall.
All the King's horses and all the King's men
Cannot put Humpty Dumpty together again.

This is perhaps the most dignified of all the statements of the second law of thermodynamics, inasmuch as it omits all nonsense about *direct repair* and refers at once to external means.

denser. Fig. 143 is a diagram for fixing in the reader's mind the various temperatures and quantities of heat and work which are involved in the operation of an engine. The engine is represented by *EE*. The boiler is at temperature T_1 and the condenser is at temperature T_2; H_1 is the quantity of heat taken from the boiler in a given time, W is the work developed by the engine in the given time over and above the work required to drive the feed water pump,* and H_2 is the quantity of heat delivered to the condenser in the given time.

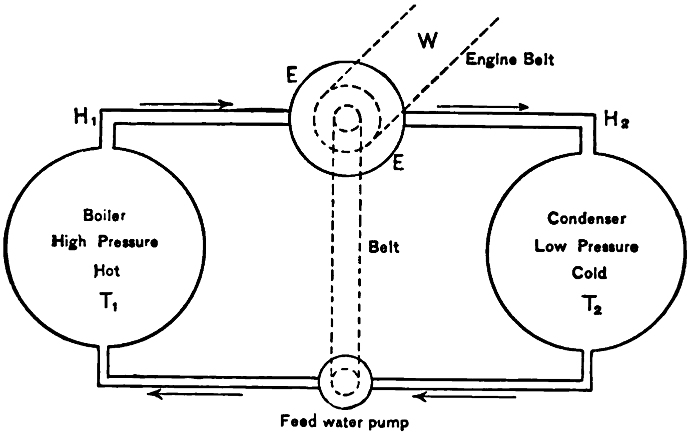

Fig. 143.

According to the first law of thermodynamics the work W done by an engine must be equal to the heat $(H_1 - H_2)$ which has disappeared. That is

$$W = H_1 - H_2 \tag{153}$$

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* The increase of pressure produced by what corresponds to the feed-water pump in the steam engine may under certain conditions cause a rise of temperature of any working fluid from condenser temperature to boiler temperature. This is not ordinarily the case in the steam engine, however. The sweeping process which would result if cool water were pumped directly into the boiler from the condenser may be obviated as nearly as may be desired by using a feed water heater in which live steam flows *from* the boiler through a pipe which is surrounded by the cool feed water flowing *towards* the boiler; the condensed water from this steam being trapped into the feed water duct and returned to the boiler.

Now, all the heat used in the engine comes from the region at temperature T_1, and the net result of the operation of the engine is: (*a*) to convert the quantity $W [= H_1 - H_2]$ of heat from temperature T_1 into work and (*b*) to transfer the quantity H_2 of heat from temperature T_1 to temperature T_2. The result (*a*) involves an amount of regeneration which is proportional to W, temperatures being given; this regeneration may, therefore, be represented by mW where m is a constant *depending only on the temperature* T_1. The result (*b*) involves an amount of degeneration which is proportional to H_2, temperatures being given; this degeneration may therefore be represented by nH_2 where n is a constant *depending only on the temperatures* T_1 *and* T_2.

If the engine is reversible we must have

$$mW = nH_2$$

or, using the value $(H_1 - W_2)$ for H_2 and solving for W, we have

$$W = \frac{n}{m + n} H_1 \tag{i}$$

in which the fraction $n/(m + n)$ depends upon T_1 and T_2, only, irrespective of the kind of engine and of the physical properties of the fluid used in the engine.

The fractional part $n/(m + n)$ of the heat H_1 which the engine converts into work is called the *efficiency* of the engine, and from equation (i) it follows that *all reversible engines have the same efficiency for given values of the temperatures* T_1 *and* T_2.

If the operation of the engine involves sweeping processes of any kind then the degeneration nH_2 exceeds the regeneration mW or

$$mW < nH_2$$

or, using the value $(H_1 - W)$ for H_2, and solving for W, we have:

$$W < \frac{n}{m + n} \cdot H_1 \tag{ii}$$

in which n and m have the same values as in equation (i). Therefore comparing equation (ii) with equation (i) it follows that *any irreversible engine (an engine which involves sweeping processes in its operation) working between given temperatures* T_1 *and* T_2 *has less efficiency than a reversible engine working between the same temperatures.*

The *refrigerating machine*, for example the ammonia refrigerator, is a machine which determines a process in which heat is transferred from a cool region to a hot region, and the thermodynamic regeneration represented thereby is compensated by the degeneration into heat of the work used in driving the machine.

Fig. 144 is a diagram for fixing in the reader's mind the various temperatures and quantities of heat and work which are

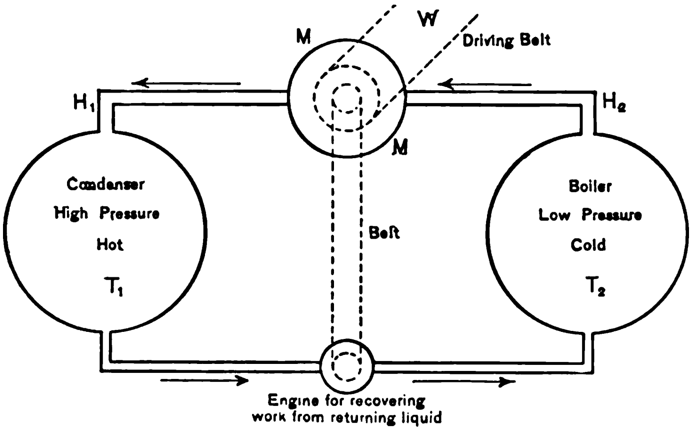

Fig. 144.

involved in the operation of the machine. The machine is represented by MM. The chamber in which evaporation takes place (the boiler) is at low temperature T_2, and the chamber in which condensation takes place (the condenser) is at high temperature T_1, H_2 is the quantity of heat taken from the cold region in a given time, W is the net work required to drive the machine during the given time, and H_1 is the heat delivered to the hot region during the given time.

According to the first law of thermodynamics the work W done in driving the machine must be equal to the heat which has been created in the operation of the machine. That is

$$W = H_1 - H_2 \tag{153) bis}$$

The net result of the operation of the machine is: (*a*) to convert the quantity W of work into heat at temperature T_1, and (*b*) to transfer the quantity H_2 of heat from temperature T_2 to temperature T_1. The result (*a*) involves an amount of degeneration which is proportional to W, temperatures being given; this degeneration may therefore be represented by mW where m is a constant depending only on the temperature T_1. The result (*b*) involves an amount of regeneration which is proportional to H_2, temperatures being given; this regeneration may therefore be represented by nH_2, where n is a constant depending only on the temperatures T_1 and T_2.

If all the processes involved in the operation of the refrigerating machine are reversible then we must have

$$mW = nH_2$$

or

$$H_2 = \frac{m}{n} W \tag{iii}$$

The number of units of heat transferred from T_2 to T_1 per unit of work spent in driving the machine, which is equal to m/n, is called the efficiency of the refrigerating machine. From equation (iii) it follows that *the efficiency of all reversible refrigerating machines is the same for given values of* T_1 *and* T_2.

If the operation of the refrigerating machine involves sweeping processes of any kind then the degeneration mW is greater than the regeneration nH_2 or

$$mW > nH_2$$

or

$$H_2 < \frac{m}{n} W \tag{iv}$$

in which m and n have the same values as in equation (iii). Comparing (iv) with (iii) it follows that *the efficiency of any irreversible refrigerating machine is less than the efficiency of a reversible refrigerating machine working between the same temperatures.*

Remark. — An engine driven backwards is a refrigerating machine, and a refrigerating machine allowed to run backwards is an engine.

280. Thermodynamic definition of the ratio of two temperatures.—A provisional definition of the ratio of two temperatures was given in Art. 224. Now, according to article 279 the ratio H_1/H_2 of the heat taken from the boiler to the heat given to the condenser by any reversible engine working between given temperatures T_1 and T_2 is invariable. The invariable value of this ratio furnishes the most satisfactory basis for the definition of the ratio of two temperatures, satisfactory because independent of the thermal properties of any particular substance. Accordingly, in thermodynamics, the ratio of two temperatures is defined by the equation

$$\frac{T_1}{T_2} = \frac{H_1}{H_2} \tag{154}$$

This definition of the ratio of two temperatures would agree exactly with the provisional definition given in Art. 224, and with temperature ratios as measured by the air-thermometer if air were an ideal perfect gas. This matter is discussed in detail in appendix A.

281. Entropy. Measure of thermodynamic degeneration. The statement of the second law of thermodynamics can scarcely be looked upon as complete until a precise and complete numerical measure of thermodynamic degeneration has been established. This numerical measure of thermodynamic degeneration is called entropy. This numerical measure of thermodynamic degeneration (entropy) may be completely established by considering in detail the action of the ideal engine as outlined in article 279 together with the thermodynamic definition of temperature ratios as given in article 280.

Referring to article 279 we may write the expression for the regeneration mW in the form $f(T_1)\cdot W$ inasmuch as m is a function of T_1 only.

The degeneration nH_2 may be written

$$[f(T_2)-f(T_1)]H_2$$

inasmuch as the degeneration associated with the transfer of the heat H_2 from T_1 to T_2 may be thought of as (*a*) the regeneration of H_2 from temperature T_1 to work, and (*b*) the degeneration of this resulting work to heat at temperature T_2; in which case the regeneration (*a*) is $f(T_1)\cdot H_2$ and the degeneration (*b*) is $f(T_2)\cdot H_2$.

Therefore the equation

$$mW=nH_2$$

may be written

$$f(T_1)\cdot W=[f(T_2)-f(T_1)]H_2$$

Using equation (153) and equation (154) we have

$$\frac{f(T_2)-f(T_1)}{f(T_1)}=\frac{T_1-T_2}{T_2}$$

Dividing numerator and denominator of the right hand member by T_1T_2 we have:

$$\frac{f(T_2)-f(T_1)}{f(T_1)}=\frac{\frac{1}{T_2}-\frac{1}{T_1}}{\frac{1}{T_1}}$$

Therefore, since T_1 and T_2 are independent of each other we have

$$f(T_1)=\frac{1}{T_1}\quad\text{and}\quad f(T_2)=\frac{1}{T_2}$$

That is to say, the thermodynamic degeneration associated with the conversion of an amount of work W into heat at temperature T_1 is equal to W/T_1, and the thermodynamic degeneration associated with the transfer of an amount of heat H_2 from temperature T_1 to temperature T_2 is

$$\frac{H_2}{T_2}-\frac{H_2}{T_1}$$

Clausius's derivation of the numerical measure of entropy is based upon the idea that degeneration and regeneration are balanced in a reversible process. In this derivation it is necessary to consider a cyclic process (reversible) in order that no outstanding change of state may be left as a result of the process, so that one need consider only the exchange of work and heat between the substance which is being studied and external substances, and in particular one is not under the necessity of considering the amount of degeneration or regeneration involved in the change of state of a particular substance.

Consider a fluid, which, starting from the state *P*, Fig. 145, is made to pass through a reversible cyclic process and return to the state *P* by any combination whatever of slow heating and cooling, expanding and compressing. The closed curve represents the cyclic process, and the moving point *Q* as it moves along the process curve represents the changing state of the fluid.

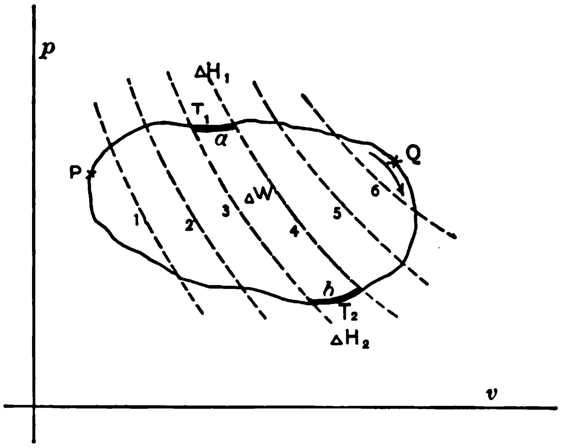

FIG. 145.

A clear idea of the external actions which take place may be obtained by drawing a series of adiabatic or isentropic lines (an isentropic line represents the variation of pressure with volume when the fluid neither gives off nor receives heat). The fluid is receiving heat while *Q* is crossing isentropic lines from small numbers to large numbers in Fig. 145, and giving off heat while *Q* is crossing isentropic lines from large numbers to small numbers. The fluid is expanding and doing external work when *Q* is moving to the right, and contracting and having work done upon it when *Q* is moving to the left. The fluid is in general at high temperatures for those positions of *Q* where *p* and *v* are both large, and at low temperatures for those positions of *Q* where *p* and *v* are both small.

Consider two portions, a and b, of the given process curve which lie between a pair of isentropic lines. Let T_1 be the high temperature of the fluid when Q is passing along a, and let dH_1 be the amount of heat taken in by the fluid while Q is passing along a. Let T_2 be the low temperature of the fluid when Q is passing along b, and let dH_2 be the amount of heat given off by the fluid while Q is passing along b.

Consider the reversible cyclic process which is represented by the two portions a and b of the given process curve, together with the isentropic lines between which a and b lie. We will call this cyclic process an elementary cyclic process to distinguish it from the given process. The net result of the elementary cyclic process would be the taking in of the quantity dH_1 of heat at T_1, the conversion of a definite fraction dW of this heat into work and the giving off of the remainder, dH_2, of the heat at temperature T_2. Therefore, according to Arts. 279 and 280 we have

$$\frac{dH_1}{dH_2} = \frac{T_1}{T_2} \tag{i}$$

Now, dH_1 is heat received by the fluid, and dH_2 is heat given off by the fluid, and one or the other should be considered as negative, say dH_2, then equation (i) should be written :

$$\frac{dH_1}{dH_2} = -\frac{T_1}{T_2} \tag{ii}$$

or

$$\frac{dH_1}{T_1} + \frac{dH_2}{T_2} = 0 \tag{iii}$$

The whole of the given cyclic process may be broken up into pairs of corresponding parts like a and b, so that the entire heat taken in and given out by the fluid during the given process consists of parts which correspond in pairs like dH_1 and dH_2, and each pair of heat parts satisfies an equation like (iii). Therefore, *the sum of all quotients obtained by dividing the heat taken in by a fluid at each step of any reversible cyclic process by the absolute temperature of the fluid at the step is equal to zero.* That is

$$\Sigma \frac{dH}{T} = 0 \tag{155}$$

for any reversible cyclic process.

Consider two states of thermal equilibrium of a fluid represented by the points A and B, Fig. 146. Let the lines a and b represent any two different reversible processes leading from A to B. Then process a together with process b *reversed* constitutes a cyclic process starting from

A and returning to A. Therefore the sum $\Sigma dH/T$ is equal to zero when it is extended over process a and over process b *reversed*. Putting

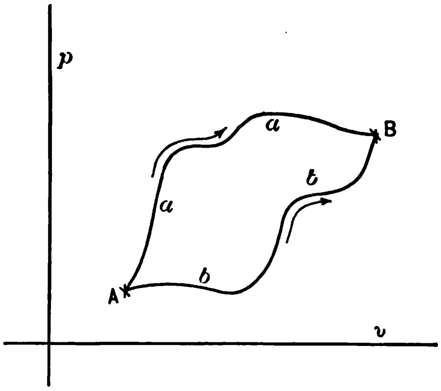

Fig. 146.

his into symbolic form we have:

$$\sum_a \frac{dH}{T} + \sum_{-b} \frac{dH}{T} = 0 \qquad \text{(iv)}$$

in which the subscript a indicates that the first summation is extended over process a, and the subscript $-b$ indicates that the second summation is extended over process b *reversed*.

But

$$\sum_{-b} \frac{dH}{T} = -\sum_b \frac{dH}{T} \qquad \text{(v)}$$

That is, the summation $\Sigma dH/T$ extended over the process b *reversed* is equal but opposite in sign to the value of this summation $\Sigma dH/T$ extended over the process b not reversed, that is, when process b leads from state A to state B.

Substituting the vatue of $\sum_{-b} dH/T$ from equation (v) in equation (iv) we have

$$\sum_a \frac{dH}{T} = \sum_b \frac{dH}{T} \qquad (156)$$

That is, *the sum* ($\Sigma dH/T$) *has the same value for any two, and therefore, for all reversible processes which lead from one given state of thermal equilibrium* A *to another given state of thermal equilibrium* B.

If the state B is one which can be reached from state A by a sweeping process, then the sum ($\Sigma dH/T$) extended over a reversible process leading from A to B is *positive* in value. Therefore the value of the sum $(\Sigma dH/T)$ extended over any reversible process leading from state A to state B of a substance *may be used as a measure of the thermodynamic degeneration* which is associated with the change of the substance from state A to state B. This sum is called the *increase of entropy* of the substance. When the sum $(\Sigma dH/T)$ is negative it measures a thermodynamic regeneration and is called a *decrease of entropy*.

Examples. — A gas is allowed to sweep through an orifice and increase in volume from v to V with imperceptible change of temperature. The same gas is then expanded slowly in a cylinder from volume v to volume V without change of temperature. To prevent change of temperature heat must be given to the gas at each step of the expansion (dH positive), and therefore the sum $(\Sigma dH/T)$ is positive.

A gas is heated at constant volume by the degeneration of work into heat. The same result may be accomplished reversibly by heating the gas slowly on a stove. In this latter process dH is positive at each step and the sum $(\Sigma dH/T)$ extended over the slow heating process is positive.

The creation of an external compensating degeneration (decrease of entropy) when the effect of a sweeping process is repaired by a reversible process may be expressed by the entropy change of external substances. Thus in each of the above examples the reversible process involves the taking of heat from external substances so that the sum $(\Sigma dH/T)$ is negative as applied to the external changes which are involved in the reversible processes mentioned, or in other words, external substances suffer an increase of entropy when a given substance has its entropy decreased by a reversible process.

In general the thermodynamic degeneration associated with a sweeping process can be represented as an increase of entropy (summation of dH/T as above explained) only by devising a reversible process which produces the same change as the given sweep so far as the substance under consideration is concerned. In the case of steady sweeps, however, it is not *necessary* to devise a reversible process for producing the same result in order to represent the result of the sweep as an increase of entropy. The entropy increase which is associated with a steady sweep may be derived from a consideration of the reversible processes which always accompany a steady sweep, using Clausius's summation $\Sigma dH/T$, as follows:

Consider the *slow* flow of heat from a body A at temperature T_1 to a body B at temperature T_2. The transfer of heat being slow, the cooling of A and the heating of B are reversible processes, A being at each instant in thermal equilibrium, and B being at each instant in thermal equilibrium.

While an amount of heat H is transferred the decrease of entropy of body A is $\Sigma dH/T = H/T_1$ and the increase of entropy of body B is $\Sigma dH/T = H/T_2$, so that the net increase of entropy due to the steady sweep is $(H/T_2 - H/T_1)$.

Consider a fine wire submerged in a large vessel of water at temperature T, heat being slowly generated in the wire by an electric current. Then the water will be at each instant in thermal equilibrium, that is, the heating of the water will be a reversible process to which Clausius's summation may be applied. Thus, while the water receives an amount of heat W (measured in terms of the work lost in the wire), the value of $\Sigma dW/T$ will be W/T, which is the increase of entropy of the water. In this case there is no decrease of entropy anywhere; so that W/T measures the thermodynamic degeneration involved in the conversion of the work W into heat at temperature T.

Absolute values of entropy. — Entropy changes or entropy differences only have real physical significance. However, a certain state of a substance may be arbitrarily chosen as a zero state or reference state and the absolute value of the entropy of the substance in any other given state may be defined as the value of Clausius's summation extended over any reversible process leading from the zero state to the given state. This is equivalent to assigning arbitrarily the value zero to the entropy of the substance in the zero state.

The entropy of a substance in a given state is proportional to the mass of the substance, for doubling the mass will double the value of dH for each step of any reversible process. Entropy is, of course, expressed in *units of heat per degree of temperature.*

282. Efficiencies of reversible engines and refrigerating machines. The actual numerical efficiencies of reversible engines and refrigerating machines may be arrived at with the help of the definition of temperature ratios given in article 280.

Engines. The work done by an engine is given by the equation

$$W = H_1 - H_2 \qquad (153)\text{ bis}$$

If the engine is reversible we have, also,

$$\frac{H_1}{H_2} = \frac{T_1}{T_2} \qquad (154)\text{ bis}$$

The efficiency of an engine is equal to the ratio W/H_1, which is the fractional part of the heat H_1 that is converted into work. Therefore, using the value of W from equation (153) we have

$$\frac{W}{H_1}=\frac{H_1-H_2}{H_1}$$

or

$$\frac{W}{H_1}=1-\frac{H_2}{H_1}$$

whence from equation (154) we have

$$\frac{W}{H_1}=1-\frac{T_2}{T_1} \tag{157}$$

in which W/H_1 is the efficiency of a reversible engine working between the temperatures T_1 and T_2. Thus a reversible engine working between a boiler temperature of 175° C. ($T_1=448°$ absolute), and a condenser temperature of 80° C. ($T_2=353°$ absolute) would have an efficiency of $(1-353/448)$ or about 21 per cent. That is, a reversible engine working between the given temperatures would convert 21 per cent. of the heat supplied to it into work and deliver 79 per cent. of the heat to the condenser. Any actual engine working between the given temperatures would have an efficiency less than this.

Refrigerating machines. The work done in driving a refrigerating machine is given by the equation

$$W=H_1-H_2 \tag{153)bis}$$

If the machine is reversible we have

$$\frac{H_1}{H_2}=\frac{T_1}{T_2} \tag{154)bis}$$

The efficiency of a refrigerating machine is equal to the ratio H_2/W, or the number of units of heat extracted from the cold region per unit of work done in driving the machine. Therefore, using the value of W from equation (153) we have

$$\frac{H_2}{W}=\frac{H_2}{H_1-H_2}$$

whence from equation (154) we have

$$\frac{H_2}{W}=\frac{T_2}{T_1-T_2} \tag{158}$$

in which H_2/W is the efficiency of a reversible refrigerating machine which extracts heat from a cold region at temperature T_2 and delivers heat to a hot region at temperature T_1. Thus a reversible machine working between a temperature of — 30° C. ($T_2 = 243°$ absolute) and a temperature of 60° C. ($T_1 = 333°$ absolute) would have an efficiency of $243 \div (333 - 243)$ or about 270 per cent. That is, a reversible refrigerating machine working between the given temperatures would extract 270 units of heat from the cold region for every 100 units of work spent in driving the machine. Any actual machine working between the given temperatures would have an efficiency less than this.

283. Conditions which limit the efficiency of engines in practice. In order that the efficiency of an engine may be as great as possible the ratio T_1/T_2 of initial temperature T_1 to final temperature T_2 of the working fluid must be as great as possible, and sweeping processes must be obviated as much as possible in the operation of the engine.

The ratio of initial to final temperature of the expanding steam or gas in an engine is determined by the ratio of initial to final volume according to equation (149) which applies accurately to a perfect gas and approximately to steam;* the expansion being of course approximately isentropic. In order that moderately small cylinders † may be used for the development of a given amount of power the initial pressure of the steam or gas must be high; and in order that the final temperature may not be lower than atmospheric or available condenser-water temperatures the initial temperature must be high. The highest boiler temperature ordinarily used in practice with steam engines is about 200°C. and the lowest condenser temperature is about 50°C. In the gas engine the initial temperature is the temperature of the mixture of air and gas immediately after the explosion, which may be from 1200° to 1700° absolute or more and the temperature is reduced by expansion to perhaps half of this (absolute). The mixture of gas and air in a gas engine is always compressed before the explosion for reasons briefly stated above.

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* The value of κ for water vapor is about 1.28.

† The reason for using small cylinders is partly on account of their cheapness, and partly on account of the great radiation of heat from large cylinders

The irreversible processes undergone by the steam in an engine are as follows:

(*a*) *Wire drawing.* — If the pipes and passages traversed by the steam from the boiler to the engine are small, the pressure in the cylinder with open ports will be lower than boiler pressure, so that the entering steam passes from a region of high pressure into a region of lower pressure. Also, as the cut-off valve closes, steam will rush into the cylinder through a narrowing aperture. To provide against loss of efficiency from this cause, the pipes must be of ample size and the cut-off must be very quick.

(*b*) *Radiation.* — The cooling of pipes and cylinder by the giving of heat to surrounding cooler bodies is a sweeping process, and is to be obviated as much as possible by covering the cylinder with a thick coating of porous insulating material.

(*c*) *Cylinder condensation.* — As one charge of steam in the cylinder expands it cools, and cools the cylinder and piston, so that when steam is next admitted it heats cylinder and piston up again and is itself cooled. This cannot be prevented, but its evil effects can be in large part avoided by having a series of cylinders (a compound engine), so that the range of pressure and the range of temperature in each may not be great; and by providing separate passages for the ingress and egress of steam.

(*d*) *Effect of high piston velocity.* — If piston speed is too great, the pressure of the expanding steam becomes ineffective because the portions of the steam near the moving piston are expanded and cooled before the more remote parts of the steam are affected.

(*e*) *Puffing.* — When the steam at the end of a stroke is still under considerable pressure it rushes through the exhaust port as a sharp puff. This is to be avoided by sufficiently reducing the steam pressure by expansion.

284. Application of thermodynamics to general physics and to chemistry. The only application of thermodynamics which is given in this chapter is to the engine and to the refrigerating machine. Thermodynamics has many other important applications to general physics and to chemistry, and the student who does not know a little of these applications has a very narrow view of thermodynamics which, as a matter of fact, is, next to mechanics, the most fundamental and the most far reaching branch of physics. Indeed physics is divided into two fundamental divisions, namely, mechanics on the one hand and thermodynamics on the other hand.

Some further applications of thermodynamics are given in appendix A.

CHAPTER XIII.

TRANSFER OF HEAT.

285. Conduction ; convection ; radiation. There are three quite distinct processes * by means of which heat is transferred from one place to another.

Conduction. — Heat may be transferred in a substance by being handed on from one part of the substance to the next part beyond. This mode of transfer is no doubt effected by invisible molecular motion. A hot region in a substance is in a state of violent molecular commotion and this commotion spreads into the adjacent cooler portions of the substance. This spreading of molecular commotion is called a flow of heat, and this mode of heat flow is called *heat conduction.*

If one end of a metal rod is held in a flame heat is transferred along the rod and the whole rod becomes more or less heated. An idea of the amount of heat which flows along a rod may be obtained by keeping one end of the rod at a high temperature in a flame, keeping the other end at a low temperature in an ice-bath, and preventing as much as possible the loss of heat through the sides of the rod by means of a covering of wool. The amount of ice melted in a given time measures the quantity of heat transferred during that time.

A substance which transfers heat rapidly by conduction is said to be a good heat conductor ; a substance which transfers heat very slowly by conduction is said to be a poor heat conductor or a heat insulator. The metals are good heat conductors. Porous substances such as saw-dust and wool are good heat insulators. Glass is a poor heat conductor. One end of a short glass rod may be held in the fingers while the other end is melted in a flame.

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* This term is not here used in the narrow sense defined in article 269.

Convection. — Heat may be transferred from one place to another by the flow of a hot fluid. Thus in a steam-heating plant, heat is carried from the furnace to the various parts of a building by the flow of steam. Great quantities of heat are carried by winds and by ocean currents from one place on the earth to another. This mode of heat transfer is called *convection.* Generally the movement of a fluid by virtue of which heat is transferred by convection is caused by the heat itself. Thus winds are caused by the unequal heating of different parts of the earth by the sun; the current of air which rises from a hot stove and spreads out in the upper part of a room is caused by the expansion of the air next the stove as it is heated; this expanded air, being lighter than the surrounding cool air, is buoyed up by the surrounding cool air which flows in to take its place.

Radiation. — Heat is transferred from a hot to a cold substance through intervening space even when this space is entirely devoid of ordinary matter. This mode of transfer of heat is exemplified in the transfer of heat and light to the earth from the sun and it is called *radiation.* The transfer of heat by radiation is effected by wave motion exactly similar in general character to the wave motion which constitutes light, these waves being transmitted by a medium, the ether, which fills all space. The molecular commotion in a hot body produces a commotion in the immediately adjacent ether, this commotion spreads out in all directions as a wave disturbance and when these waves impinge on a cool body they produce molecular commotion in it. This wave commotion in the ether is called *radiant heat,* a term which embraces light as well as heat. Radiant heat is treated in Volume III., Chapter XII.

Generally the transfer of heat takes place by all three of the above processes simultaneously. Thus heat is distributed throughout a room from a hot stove partly by radiation, for one can feel with the hand the heat of the stove at a distance even though the air next the hand is quite cold; by convective currents of air; and by conduction. The last process, however, is almost inappreciable in air inasmuch as air is a very poor heat conductor.

286. Temperature gradient; Fourrier's law. Thermal conductivity — The difference in temperature of a substance at two points divided by the distance between the points is called the *temperature gradient* between the points. Thus one side of a wall is at 0° C., the other side is at 75° C., the wall is 25 cm. thick, and the temperature gradient through the wall is 75°/25 cm., or three degrees per centimeter.

Let T be the temperature of a substance at a point A and $T + \Delta T$ its temperature at an adjacent point B distant Δx from A. Then $\Delta T/\Delta x$ or dT/dx is the temperature gradient between A and B.

The conductive flow of heat in a substance is always associated with a temperature gradient; the temperature of the substance falls off in the direction of the flow of heat.

Consider a metal rod, of sectional area q, along which heat is flowing by conduction. Let dT/dx be the temperature gradient along the rod, and let $\dot{H}$ be the number of units of heat per second flowing past a given section of the rod, then $\dot{H}$ is proportional to q and to dT/dx (Fourrier's law). We may therefore write:

$$\dot{H} = Kq\frac{dT}{dx} \tag{159}$$

in which K is a proportionality factor which has a definite value for a given substance. This quantity K is called the *thermal conductivity* * of the substance.

Remark.—The thermal conductivity of a pure metal is approximately proportional to the reciprocal of absolute temperature. Thus the thermal conductivity of pure copper is, say, 0.90 at 0°C. and it is about 273/373 × 0.90 at 100°C.

287. Flow of heat through a wall.—Consider a wall or slab of a substance of thickness d, the faces of the wall being kept at temperatures T' and T'' respectively. Let q be the area of

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* For discussion of methods for experimentally determining thermal conductivity, see Preston's "Theory of Heat," pp. 505–572.

TABLE

THERMAL CONDUCTIVITY (K); CALORIES CM.$^{-1}$ SEC.$^{-1}$ PER °C.

Substances.	K.	Substances.	K.
Aluminum	0.34 at 0°	Cork	0.0007
Antimony	0.04 at 0°	Glass	0.0013 to .0018 at 15°
Bismuth	0.01 at 0°	Ice	0.0056
Carbon	0.15 to 0.20 at 0°	Lava	0.00008 at 0°
Copper	0.90 to 1.10 at 0°	Marble	0.0017 at 0°
Iron	0.15 to 0.20 at 0°	Paraffine	0.00014 at 0°
Lead	0.07 at 7°	Vulcanite (hard rubber)	0.00009 at 0°
Magnesium	0.37 at 50°		
Silver	1.09 at 0°	Alcohol (C_2H_6O)	0.0004 at 5_0
Tin	0.14 at 0°	Ether ($C_4H_{10}O$)	0.0004 at 5°
Zinc	0.30 at 0°	Mercury	0.0148 at 0°
		Olive oil	0.0004 at 5°
		Water	0.0012 at 0°

the wall, and K the thermal conductivity of the substance of the wall. Then $(T' - T'')/d$ is the temperature gradient through the wall, which substituted for dT/dx in equation (159) gives:

$$\dot{H} = \frac{Kq(T' - T'')}{d} \tag{160}$$

in which $\dot{H}$ is the number of units of heat flowing through the wall per second. When the values given in the above table are used for K, when d is expressed in centimeters, q in square centimeters, and T' and T'' are expressed in the centigrade scale, then $\dot{H}$ is expressed in calories per second.

288. Emission of heat.*—A hot body suspended in the air gives off heat to the air and to surrounding bodies by convection, by radiation, and to a very slight extent by conduction. The body is said to *emit* heat.

The rate at which a body emits heat depends upon its excess of temperature above its surroundings, upon the extent and character of the surface of the body, upon the nature of the surrounding gas, upon the freedom of motion of the surrounding gas, and upon the nature of surrounding bodies, and there is no general law of the emission of heat. There are cases however in which it is necessary to make approximate calculations of the amount of heat emitted by a body. Thus in accurate work with the water calorimeter the error due to radiation must be approximately calculated and applied as a correction. In the design of heating plants for buildings the heat emitted by the steam heating coils must be approximately calculated so that these coils may be properly designed for the service required. Such calculations are based upon a simple approximate law of emission discovered by Newton:

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*This term as used in Chapter XII. of Volume III. refers to the emission of heat by a body by radiation only. Here the term means the giving off of heat by a body to the surrounding air and to surrounding bodies by conduction, by convection, and by radiation.

Newton's law of emission. — The rate $\dot{H}$ at which a body emits heat is very nearly proportional to the excess t of its temperature above the temperature of its surroundings and roughly proportional to the extent of surface q of the body. That is

$$\dot{H} = eqt \tag{161}$$

where e is a proportionality factor which depends chiefly upon the character of the surface of the body. It is called the *emissivity* of the surface.

A rough blackened surface emits heat more rapidly than a bright polished surface. The value of e for a given character of surface varies considerably with the freedom of motion of the surrounding air and ease of access of the air to the hot surface, for this determines the extent to which convection currents of air are established as the body gives off heat.

For these reasons the value of e must be determined under conditions which are in a general way similar to the conditions to which equation (161) is afterwards to be applied. Thus, if one is to calculate the heat loss of a water calorimeter by means of equation (161) the experiments for the determination of e should be carried out on the identical calorimeter for which the calculations are afterwards made; if one is to calculate the heat given off by steam coils by means of equation (161) the value of e should be determined by experiments on steam coils.

PROBLEMS.

CHAPTER I.

MEASUREMENT OF LENGTH, TIME, AND MASS.

1. From the data given in Art. 1 calculate the number of centimeters in one inch, the number of centimeters in one foot, the number of square centimeters in one square inch, the number of cubic centimeters in one cubic inch, the number of cubic centimeters in one cubic foot, and the number of cubic centimeters in one gallon (231 cubic inches).*

2. A scale has half inch divisions, and a vernier to be used with this scale is $7\frac{1}{2}$ inches long and it is divided into sixteen equal parts. To what fractional part of an inch does the vernier read?

3. The divisions on the circle of a surveyor's transit are $\frac{1}{3}$ of a degree. A vernier is to be made for use with this circle and to read to $10''$. What is the length of the vernier and what is the number of divisions upon it?

4. The density of alcohol is 6.35 pounds per gallon. What is its density in pounds per cubic inch? What is its density in grams per cubic centimeter?

5. The density of iron is 7.8 grams per cubic centimeter. What is its density in pounds per cubic inch? What is the weight of an iron bar 30 feet long and $5\frac{1}{2}$ square inches sectional area?

6. A bottle weighs 50.62 grams empty, and 288.93 grams full of water at 21° C. What is the cubic contents of the bottle? The bottle full of oil at 21° C. weighs 239.2 grams. What is the sp. gr.

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* The results of problems which are marked with an asterisk should be recorded for future use.

of the oil? What is the density of the oil? Neglect in this and the following problems the effect of buoyant force of the air.

See table of densities of water at various temperatures. See index.

7. A piece of lead weighs 233.60 grams in air and 212.9 grams in water at 20° C. What is the specific gravity and the density of lead at 20° C.?

8. A piece of glass weighs 260.7 grams in air and 153.8 grams in water at 20° C. The same piece of glass weighs 92.2 grams in H_2SO_4 at 20° C. What is the sp. gr. of H_2SO_4 at 20° C.?

9. The distance along a hydrometer stem from the water mark to the mark to which the instrument sinks in kerosene (sp. gr. .79) is 9.62 cm. Calculate the distance from the water mark to the marks to which the instrument would sink in a 20% solution of alcohol, in a 40% solution of alcohol, in a 60% solution of alcohol, in an 80% solution of alcohol, and in pure alcohol. The specific gravities of these solutions are as follows: 20% = .975; 40% = .951; 60% = .913; 80% = .863; 100% = .794.

10. A clock with a pendulum which is supposed to beat seconds reads $12^h, 0^m, 16^s.8$, at mean noon on one day and it reads $12^h, 1^m, 2^s.3$, at mean noon ten days later. What number of beats does the pendulum of the clock make in a mean solar day? What is the true value in hours, minutes, and seconds of an interval of time at the beginning of which the above clock reads $5^h, 6^m, 10^s.2$, and at the end of which the clock reads $8^h, 33^m, 56^s.7$?

CHAPTER II.

PHYSICAL QUANTITY.

11. A table top is 10 feet long and 50 inches wide. Find its area in inch-feet, and explain the result.

12. A body has a mass of 60 pounds and a volume of 2 gallons. Find its density without reducing data in any way.

13. A man travels at a velocity of 6 feet per second; how far does he travel in two hours? Find the result without reducing the data in any way. Explain the *foot-hour per second* as a unit of length.

14. If 500 grams of water leak out of a pail in 26 seconds, what is the average rate of leak?

A man earns $27.50 in 8½ days. What is the average rate at which he earns money?

During 28 seconds the velocity of a train increases from zero to 12 feet per second. What is the average rate of increase of velocity.

15. A train gains a speed of 32 miles per hour in 80 seconds. Find its average acceleration in miles per hour per second.

16. A pole 22 feet long is dragged sidewise over a field at a velocity of 8 feet per second. At what rate does the pole sweep over area?

17. A prism has a base of 25 square cm., its height is increasing at the rate of 5 cm. per second. How fast is its volume increasing?

18. The slope of a hill falls 60 feet in a horizontal distance of 270 feet. What is the grade?

19. One side of a brick wall is at a temperature of 0° C. and the other side is at a temperature of 23° C. The wall is 30.5

cm. thick. What is the average temperature gradient through the wall?

20. At a given point in a water pipe the water pressure is 110 lbs. per square inch. Twenty-two feet from this point the pressure is 75 pounds per square inch. What is the average pressure gradient along the pipe?

CHAPTER III.

FORCE; LAWS OF MOTION.

21. A gun which produces a projectile velocity of 200 feet per second is mounted aboard a car with its barrel at right angles to the direction of motion of the car, as shown in Fig. 21*p*. The car is travelling 50 feet per second. The sights have to be arranged at an angle θ to the gun barrel as shown. Find the value of θ so that the ball may hit any object which, at the instant of firing, is in the line of the sights.

Remark. — This problem illustrates the vector addition of velocities. The sight line must of course be in the actual direction in which the bullet is moving when it leaves the gun.

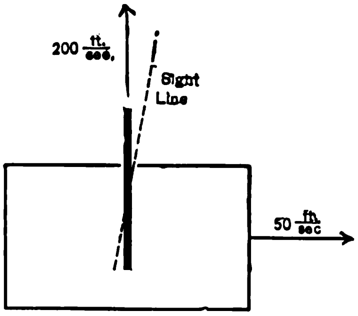

Fig. 21*p*.

22. Calculate the number of dynes in the weight of a gram and in the weight of a pound. Calculate the number of dynes in the force which will accelerate one pound mass at the rate of one foot per second per second. (This force is called the *poundal.*) Calculate the number of poundals in the weight of a pound.*

23. A train starting from rest reaches full speed of 50 miles per hour in 2½ minutes. What is the average acceleration? Express the result in terms of the foot and the second, and in terms of the centimeter and the second.

24. The above train has a mass of 350 tons (2,000 lbs.).

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*The acceleration of gravity is 980 cm. per second per second or 32 feet per second per second. One pound is equal to 453.6 grams.

What is the average pull of the locomotive in pounds weight while starting, dragging forces being ignored?

25. The above train moving at a speed of 50 miles per hour is brought to a standstill in 16 seconds by the brakes. What is the average acceleration (negative) while stopping, and what is the average retarding force in pounds weight due to the brakes?

26. An elevator reaches full speed of 8 feet per second $2\frac{1}{2}$ seconds after starting. With what average force in pounds weight does a 160-pound man push down on the floor while the elevator is starting up? The elevator is stopped (when moving up at full speed) in $1\frac{1}{2}$ seconds. With what average force in pounds weight does a 160-pound man push down on the floor while the elevator is stopping?

Remark. — In the first case the upward push of the floor on the man exceeds the weight of the man by the amount which is necessary to produce the upward acceleration; in the second case the weight of the man exceeds the upward push of the floor by the amount which is necessary to produce the downward acceleration.

27. A falling ball passes a given point at a velocity of 12 feet per second. How far below the point is the ball after 5 seconds? How far does the ball fall during the fifth second after passing the given point?

28. A heavy iron ball is tossed at a velocity of 20 feet per second in a direction 30° above the horizontal. What are its horizontal and vertical distances from the starting point after $\frac{3}{4}$ second?

Remark. — Find vertical and horizontal components of the initial velocity. The latter component remains unchanged while the vertical motion of the ball is precisely what it would be if it had no horizontal motion.

29. The mass of the moon is $\frac{1}{80}$ of the mass of the earth and its radius is 0.27 as great. How far would a body fall in one second at the surface of the moon?

Remark. — The force mg with which a body of mass m is attracted by the earth at its surface is:

$$mg = k\frac{mE}{r^2}$$

and the force mg' with which the same body is attracted by the moon at its surface is:

$$mg' = k\frac{mM}{R^2}$$

in which g is the acceleration of gravity at the earth's surface, g' is the acceleration of gravity at the moon's surface, E is the mass of the earth, M is the mass of the moon, r is the radius of the earth, and R is the radius of the moon.

30. Show from equation (8) that the dimensions of force are ml/t^2. See Art. 20.

31. Show from equation (17) that the dimensions of momentum are ml/t.

32. A car loaded with clay has a mass of 95,000 pounds. What velocity would be imparted to this car by shooting a 550-pound ball into one end of it at a velocity of 2,000 feet per second?

33. An ivory ball of which the mass is 500 grams collides with a stationary ivory ball of which the mass is 1,000 grams, the line connecting the centers of the balls being parallel to the velocity of the moving ball. Find the common velocity of both balls after their relative motion has been reduced to zero during the first half of the impact, and find the velocity of each ball after impact; specify direction of each velocity.

34. A ball weighing 550 pounds is shot from a 150,000-pound gun at a velocity of 2,500 feet per second. What is the backward velocity of the gun as the ball leaves the muzzle? Suppose the gun is allowed to move back two feet during the recoil, what is the average value of the force required to bring it to rest?

35. The mass of the earth is 80 times the mass of the moon. The distance from the center of the earth to the center of the moon is 240,000 miles. What is the distance from the center of the earth to the center of mass of the earth and moon?

36. A wheel weighing 280 pounds is out of balance. A weight of 6 pounds attached to the rim of the wheel (2 feet from the center of the wheel) brings the center of mass of wheel and weight to the center of the wheel. How far from the center of the wheel is its center of mass and in what direction with reference to the line drawn from the center of the wheel to the attached weight?

37. Three weights, *A*, *B*, and *C*, are placed upon a light horizontal board. Weight *A* has a mass of 250 grams and is placed 20 cm. north of the center of the board, weight *B* has a mass of 300 grams and is placed 15 cm. east of the center of the board, and weight *C* has a mass of 360 grams and is placed 10 cm. south of the center of the board. What are the coördinates of the center of mass of the three weights?

CHAPTER IV.

CIRCULAR AND HARMONIC MOTION; STATICS; ENERGY.

38. Derive an equation corresponding to equations (28)*a* and (28)*b* expressing in pounds weight the force necessary to constrain a particle to a circular orbit, radius of orbit being in feet, velocity of particle being in feet per second, and mass of particle in pounds.

39. An 80-ton (2,000 pounds) locomotive goes round a railway curve of which the radius is 600 feet at a velocity of 65 feet per second. With what force in pounds weight do the flanges of the wheels of the locomotive push against the outer rail, when the outer rail is not elevated?

40. The unbalanced wheel described in problem 36 is fixed upon a shaft and set rotating at a speed of 1,200 revolutions per minute. With what force in pounds weight does the shaft push upon its bearings and what is the direction of this force at each instant, ignoring forces due to gravity?

41. Calculate the proper elevation to be given to the outer rail on a railway curve of 600 feet radius for a train speed of 65 feet per second, the width of the track being 4 feet $8\frac{1}{2}$ inches.

42. Derive an equation corresponding to equation (32) expressing in pounds weight the force necessary to cause a particle to perform simple harmonic motion; mass of particle being in pounds, and displacement x being in feet.

43. Derive an equation corresponding to equation (34) expressing in *pounds weight per foot* the constant a of a spring, the mass of the vibrating ball being in pounds.

44. A force of 2 megadynes (2,000,000 dynes) deflects the end of a spring through a distance of 1.5 cm. Find the constant of the spring and state the unit in terms of which the constant is expressed. How much force would be required to deflect the end of the spring 0.5 cm., and how much to deflect the end of the spring 1.0 cm.?

45. A mass of 5 kilograms fixed to the end of a spring makes 80 complete vibrations in two minutes. Find the constant of the spring and state the unit in terms of which the constant is expressed.

46. A force of 10 pounds' weight deflects the end of a spring through a distance of ¾ inch. Find the constant of the spring in pounds weight per foot.

47. A mass of 30 pounds fixed to the end of a spring makes 75 complete vibrations in two minutes. What is the constant of the spring in pounds weight per foot?

48. A strip of spring steel has one end clamped in a vise. A side force of 20 pounds weight acting at the free end of the spring deflects the end through a distance of ¾ inch. A weight of 50 pounds mass is attached to the free end of the spring. Calculate the number of vibrations (complete) per second of the attached weight.

49.* A simple bridge truss consists of two struts and a tie rod as shown in Fig 49*p*. A weight W of 20,000 pounds is hung

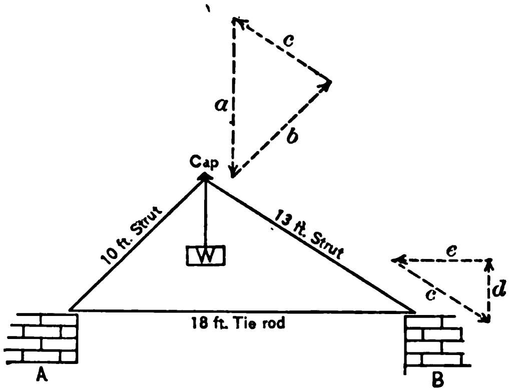

Fig. 49*p*.

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* In a short course in physics but little time should be spent upon the subject of statics.

from the cap as shown. Find the compressing force in each strut, vertical force on each abutment, and tension in the tie rod.

Remark. — The cap is a body in equilibrium so that the given force a, Fig. 49p, of 20,000 pounds is balanced by the two thrusts b and c which are parallel to the respective struts.

The thrust c of the 13-foot strut at B is balanced by the upward thrust d of the abutment, and the pull e of the tie rod.

The thrust b of the 10-foot strut is balanced in a similar manner at A.

50. A table drawer of breadth b and depth (front to back) a is pulled by a force F applied at a distance x from one corner as shown in Fig. 50p. The drawer binds at the two corners p and q and it is required to find the smallest value of x for which the drawer can be pulled out by the force F; the coefficient of friction between drawer and guides being μ.

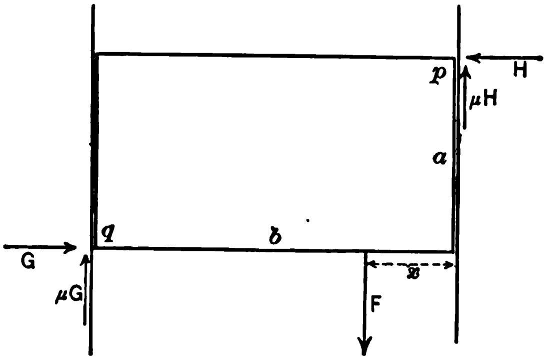

Fig. 50p.

Remark. — At p the guide exerts upon the drawer the force H and another force which cannot exceed μH. At q the guide exerts upon the drawer the force G and another force which cannot exceed μG. Now when it is just possible to pull out the drawer the various forces F, G, H and the full value of the forces μH and μG are in equilibrium. The algebraic conditions of equilibrium are:

1. That the sum of all forces to the right is equal to the sum of all the forces to the left. That is

$$G = H \qquad \text{(i)}$$

2. That the sum of all the downward forces is equal to the sum of all the upward forces. That is

$$F = \mu G + \mu H \qquad \text{(ii)}$$

3. That the sum of all the right handed torque actions about any chosen origin of moments is equal to the sum of all the left handed torque actions. Choosing the point q as the origin of moments, this condition gives

$$\overset{\text{Right handed torques.}}{F(b - x)} = \overset{\text{Left handed torques.}}{b\mu H + aH} \qquad \text{(iii)}$$

All three unknown forces F, G and H may be eliminated from these equations and the value of x determined in terms of b, a and μ.

51. Calculate the number of dyne-cm. in one pound-inch of torque and in one pound-foot of torque.*

52. Calculate the number of ergs in one-foot pound of work. Calculate the number of ergs in one watt-hour and the number of joules in one horse-power hour.*

53. Calculate the number of watts in one foot-pound per second and calculate the number of watts in one horse-power.

Remark. — In problems 51, 52, and 53 the following data, only, should be used: one foot = 30.5 cm., one pound = 453.6 grams, one gram weight = 980 dynes, and 1 horse-power = 746 watts.

54. A 165-pound man climbs up a 12-foot stair. How much work is done? Express the result in foot-pounds, in ergs, in joules, in kilowatt-hours and in horse-power hours. Power at 10 cents per kilowatt-hour costs how much per horse-power hour?

55. A 1,000 horse power boiler and engine plant costs about $70,000 complete, including land, building, boilers, engines and auxiliary apparatus such as pumps and feed water heaters. The cost of operating this plant continuously, night and day, is as follows:

Interest on investment.	5 per cent.
Depreciation	10 " "
Maintenance and repairs	4 " "
Taxes and insurance	2 " "

Labor $30 per day, 365 days in the year.

Coal $2.00 per ton.

The average demand for power is 50 per cent. of the rated power output of the plant, that is 500 horse power, and the consumption of coal is 2½ pounds per horse-power-hour. Find the cost of a horse-power-hour delivered by the engine. Ans. 0.83 cent.

The above engine will drive a 700 kilowatt dynamo, that is a dynamo capable of delivering 700 kilowatts. The cost of dynamo, station wiring and switch-board apparatus is $20,000. The average output of the dynamo is 350 kilowatts (corresponding to 500 horse power output of engine). Calculate the cost of electrical energy per kilowatt-hour at the station allowing 21 per cent. for interest, depreciation, etc., on the electrical machinery and allowing $5 per day additional for labor. Ans. 1.31 cents.

56. A horse pulls upon a plow with a force of 100 pounds weight and travels 3 miles per hour. What power is developed? Express the result in foot-pounds per second, in watts, and in horse-power.

57. A belt traveling at a velocity of 70 feet per second transmits 360 horse-power. What is the difference in the tension of the belt between the tight and loose sides in pounds weight?

58. A stream furnishes 500 cubic feet of water per second at a head of 15 feet. What power can be developed from this stream by a water wheel of which the efficiency is 60%?

59. The engines of a steamship develop 20,000 horse-power, of which 30% is represented in the forward thrust of the screw in propelling the ship at a speed of 17 miles per hour. What is the forward thrust of the screw in pounds weight?

60. A bicycle rider has a 50-foot hill to climb. What velocity must he have at starting to relieve him from the doing of one third of the work required?

61. The rim of the fly-wheel of a metal punch is 5 feet in diameter and its mass is 560 pounds. At what initial speed must the fly-wheel run in order that the punch may exert a force of 72,000 pounds through a distance of one inch and reduce the speed of the fly-wheel only 30%?

62. A steamship has a gross mass of 25,000 tons. What is the kinetic energy of the ship at a speed of 18 miles per hour? Express the result in foot-pounds.

63. A shaft transmits 100 H. P. at a speed of 250 revolutions per minute. Calculate the torque. Express the result in pound-feet and in dyne-cm.

64. Show from equation (40) that the dimensions of energy are ml^2/t^2; show from equations (41) that the dimensions of power are ml^2/t^3.

Remark. — The dimensional relations pointed out in this problem and in problems 30 and 31 are of importance in determining the unit in terms of which the result of a calculation is expressed. Aside from the question of units it is of greater importance to remember that force is *mass times acceleration* than to remember that its dimensions reduced to fundamental quantities are ml/t^2; to remember that work is *force times a parallel distance* than to remember that its dimensions are ml^2/t^2 in terms of fundamental quantities; to remember that power is *work per unit time* than to remember that its dimensions are ml^2/t^3 in terms of fundamental quantities; and so on through the whole range of derived quantity.

CHAPTER V.

MOTION OF RIGID BODIES.

65. How many radians per second in one revolution per second? In one revolution per minute? *

66. A body starts from rest and after 10 seconds it is rotating 55 revolutions per second. What is the average angular acceleration? Express the result in radians per second per second.

67. In what terms is moment of inertia expressed; (*a*) When length is expressed in centimeters and mass in grams? (*b*) When length is expressed in inches and mass in pounds? (*c*) When length is expressed in feet and mass in pounds? The unit moment of inertia in case (*a*) is the c. g. s. unit. How many c. g. s. units of moment of inertia are there in the unit involving the inch and the pound, and in the unit involving the foot and the pound? *

68. Calculate the moment of inertia of a uniform slim rod, length 31 cm. ($= l$) and mass 360 grams ($= m$), about an axis passing through the center of the rod and at right angles to the length of the rod.

(*a*) Calculate K from the formula $K = ml^2/12$.

(*b*) Calculate K approximately by multiplying the mass of each centimeter of the rod by the square of its estimated mean distance from the center of the rod.

(*c*) Calculate the radius of gyration of the rod.

69. Calculate the moment of inertia of a circular disk, radius 17 cm., mass 425 grams, about the axis of figure.

(*a*) Calculate K from the formula given in the table in Art. 93. The circular disk is of course a very short cylinder.

(*b*) Calculate the radius of gyration of the disk.

70. Calculate the moment of inertia of the rod, problem 67, about an axis passing through the end of the rod and perpendicular to the rod.

Calculate the moment of inertia of the disk, problem 68, about an axis passing through the edge of the disk parallel to the axis of figure of the disk.

71. A circular disk, 5 feet diameter, weighing 1,200 pounds is mounted upon a shaft 6 inches in diameter. The disk, set rotating at 500 revolutions per minute and left to itself, comes to rest in 75 seconds. Calculate average (negative) angular acceleration while stopping, calculate average torque acting to stop the disk, and calculate the frictional force at the circumference of the shaft. Ans. (*a*) 0.698 radians per second per second, (*b*) 81.8 pound-feet,* (*c*) 327.2 pounds.*

72. A slim rod 20 cm. long and weighing 250 grams, is suspended by a wire. The wire is attached to the middle of the rod and the rod hangs in a horizontal position. The rod, set vibrating about the wire as an axis, makes 50 complete vibrations in 10 minutes 25 seconds. What torque would be required to twist the wire through one complete turn?

73. The axis of a gyroscope points north and south and the south end of the axis rests upon a support. The gyroscope turns clockwise as seen from the north. Which way does the axis of the gyroscope sweep around the support as seen from above? Explain.

74. The armature shaft of a ship's dynamo is athwart ship and the armature is driven clockwise as seen from the port side of the vessel. Describe accurately the forces with which the bearings act upon the armature shaft as the vessel rolls. Specify the directions of these forces when the port side of the vessel is rising, and when the port side of the vessel is falling.

This question refers to the forces which arise from the rotatory motion of the armature. The port side of a vessel is on the left hand of a person facing the bow.

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* The pound is here used as a unit of force and as such should properly be called *pounds' weight.* This term is however often inconvenient in speech and the student must learn to distinguish between the pound of mass and the pound of force.

75. A high speed engine with its shaft athwart ship, makes 240 revolutions per minute. The rim of the fly-wheel has a radius of 3 feet and a mass of 600 pounds. Calculate the moment of inertia of the wheel (rim). The maximum angular velocity attained by the vessel in rolling is $\frac{1}{10}$ radian per second. Calculate the maximum torque acting on the fly-wheel shaft due to gyroscopic action and specify how the torque acts.

Remark. — The angular velocity at which the axis of a gyroscope sweeps round (in a plane) multiplied by the angular velocity of the gyroscope wheel gives the angular acceleration of the gyroscope wheel in radians per second per second and the product of this angular acceleration into the moment of inertia of the wheel gives the torque acting to make the axis sweep. When torque is to be expressed in pound-feet, moment of inertia in lb.-ft.^{2}, and angular acceleration in radians per second per second then

$$T = \frac{1}{32} K \dot{\omega}$$

76. Two balls of lead, each weighing one gram, are fixed to the ends of a horizontal rod of negligible mass. The rod is 50 cm. long, so that the moment of inertia of the two balls about the center of the rod is approximately $2 \times (25)^2$ gram $\times$ cm^2. The rod, suspended by a long fine wire, is observed to make one complete torsional vibration in 325 seconds. A very large mass of lead brought near and to one side of one of the one-gram balls is observed to hold the bar steadily one half of a degree of angle from its equilibrium position. With what force does the mass of lead pull on the one-gram ball?

Remark. — For the sake of simplicity consider only the attraction of the large mass of lead for the adjacent one-gram ball.

77. An irregularly shaped body weighing 2,800 grams is hung as a pendulum at a point which is 20 cm. distant from its center of mass (determined by balancing the body on a sharp edge). The body vibrates 90 times in two minutes (complete vibration). What is the moment of inertia of the body about the axis of suspension? What is the moment of inertia of the body about an axis through its center of mass parallel to the axis of suspension?

78. A uniform slim rod 125 cm. long is hung as a pendulum at a point distant 6 cm. from the end of the bar. Calculate its equivalent length as a pendulum. Grasping this rod at the point of suspension, how far from the hand is the point of the rod where a quick side blow on the rod would not sting the hand, the rod being assumed to be rigid.

79. A pendulum clock rated in Boston and carefully transported to Hammerfest would gain how many seconds per day? *See table in article 107.*

CHAPTER VI.

ELASTICITY.

80. A rod initially 16 feet long is stretched to a length of 16 feet 1 inch. Find the value of the stretch.

81. The three stretches of a strain are $+0.015$, $+0.025$ and -0.025. What are the semi-axes of the ellipsoid into which a sphere 10 inches in radius is distorted by this strain? Strain supposed to be homogeneous.

82. A force of 250 pounds acts across a section of which the area is $\frac{1}{4}$ square inch. What is the value of the stress on the section?

83. A square rod $2 \times 1\frac{1}{2}$ inches is subjected to a tension of 75,000 pounds. What kind of stress acts across a section of the rod and what is its value.

84. Two long strips of metal are lapped and fastened by a single rivet of which the sectional area is two square inches. The two strips are subjected to a tension of 10,000 pounds. What kind of stress acts across the middle section of the rivet and what is the value of the stress?

85. A rod 20 feet long and 1 inch in diameter is subjected to a pull of 4,000 pounds per unit of sectional area causing it to be lengthened to 20.02 feet, that is one part in a thousand, and causing it to contract to a diameter of 0.9997 inch, that is three parts in ten thousand. What is the length and what is the diameter of the wire when it is subjected to a pull of 8,000 pounds per unit sectional area?

86. A helical spring is elongated by the amount of 1.2 inches when a 4-pound weight is hung upon it. How much additional elongation is produced by 1 pound additional weight? By two pounds additional weight? By three pounds additional weight? By four pounds additional weight?

87. The middle of a long beam is depressed 2 inches by a load of 5,000 pounds. How much will it be depressed by a load of 15,000 pounds?

88. A long rod is fixed at one end and a twisting force or torque of 100 pound-inches applied at the free end causes the free end to turn through an angle of 10°. What torque would be required to turn the free end of the rod through 30°?

89. A wire 200 inches long and 0.1 inch in diameter is pulled with a force of 150 pounds. The elongation produced is $\frac{1}{2}$ inch. What is the value of the stretch modulus of the material? What does the result mean? Ans. 7,640,000 pounds per square inch. (= force per unit area which would produce *unit elongation per unit initial length*, or, which would double the length of the wire if the proportional relationship between stress and strain were true for such an extreme case of distortion.)

90. A wire five feet long and 0.06 square inch sectional area is subjected to a stretching force of 300 pounds. The stretch modulus of the material is 8,000,000 pounds per square inch. What elongation is produced?

91. The stretch modulus of a substance is 1.8×10^{12} dynes per square centimeter [$= 1/a$, equations (66)] and the value of Poisson's ratio for the substance is $\frac{1}{4}$[$= b/a$, equations (66)]. Find the values of a and b in cm^2/dyne.

92. Calculate the three pulls F, G and H of the stress which will cause an elongation of the substance mentioned in problem 91 of one part in a thousand in a given direction ($f = 0.001$), without producing any change of dimensions in directions at right angles to the given direction ($g = h = 0$).

93. A body subjected to hydrostatic pressure is decreased in length, in breadth, and in thickness by 5 parts in a thousand (initial). By how many parts per thousand (initial) is the volume reduced?

94. 2,000 cubic inches of water are reduced to 1,880 cubic inches by an hydrostatic pressure of 3,000 lbs. per square inch.

What is the value of the bulk modulus of water? What does the result mean? Ans. 300,000 pounds per square inch (= pressure which would produce unit diminution of volume per unit initial volume, or, would reduce the volume to zero if each additional unit of pressure would produce the same diminution of volume as the first unit pressure applied. As a matter of fact this is quite accurately true up to a few thousand pounds per square inch (Hooke's Law), but it is not even approximately true for excessively high pressures).

95. Calculate the compressibility of water from the answer to the previous problem and explain its meaning. Ans. $\frac{1}{300000}$ per pound-per-square-inch (= diminution-of-volume-per-unit-volume per unit pressure. That is, the volume of water is diminished one part in 300,000 for each pound per square inch of pressure).

96. A cubical piece of steel is shortened two parts in a thousand in one direction, lengthened two parts in a thousand in a direction at right angles to the first, and unchanged in dimension in the third direction, as represented in Fig. 73. What is the value of the angle $\pi/2 - \theta$ in radians and in degrees, θ being the angle shown in Fig. 73.

97. The stretch modulus of steel is 1.8×10^{12} dynes per square centimeter and the value of the slide modulus is 0.72×10^{12} dynes per square centimeter. What is the value of the bulk modulus and of Poisson's ration?

98. A steel beam is bent so that its middle line forms the arc of a circle 600 inches in radius. What is the elongation per unit length of a filament 2 inches from the middle line?

99. The stretch modulus of the steel of which the beam of the previous problem is made is 15,000,000 pounds per square inch. What is the pull (force per unit area of course) of a filament of the beam 2 inches from the middle line of the beam?

100. A rod 120 inches long is fixed at one end and the other end is turned through $\frac{1}{10}$ of a radian of angle. Consider a small portion p of the rod at a distance of 1 inch from the axis of the rod. Find the angle of shear ϕ of this small portion of metal and specify the values and directions of its three stretches. Ans. $\phi = \frac{1}{1200}$ radian. The stretch parallel to the radius of the rod is zero. The other two stretches are + one part in 2,400 and − one part in 2,400 respectively, and their directions are at right angles to the radius of the rod, making angles of $+45°$ and $-45°$ respectively with a line passing through p parallel to the axis of the rod.

101. A shaft 500 inches long and 3 inches in diameter transmitting 100 H. P. is subject to a torque of 23,100 pound-inches of torque. The slide modulus of the material of the shaft is 7,000,000 pounds per square inch. Calculate the angle through which one end of the shaft is twisted relative to the other end.

CHAPTER VII.

HYDROSTATICS AND HYDRAULICS.

102. Calculate the number of dynes per square centimeter and grams per square centimeter in one pound per square inch pressure.*

103. Calculate the total force on a piston 10 inches in diameter (piston rod 2 inches in diameter), steam pressure 125 pounds per square inch.

104. Calculate the pressure at a point 3,050 centimeters (100 feet) beneath the surface of water; value of gravity 980 centimeters per second per second, density of water one gram per cubic centimeter.

Remark. — When distance x beneath surface of water is given in *feet*, the pressure in *pounds per square inch* is given by the formula:

$$p = 0.45x$$

The physical dimensions of this factor 0.45 are of course *pounds per square inch per foot* or *pounds per* ($inch^2 \times foot$) or the density of water in pounds per unit volume where the unit volume is the volume of a parallelepiped one inch square and one foot long. See problems 11, 12, and 13.

105. Sketch the form of a dish such that the total force due to hydrostatic pressure on its bottom shall be (*a*) greater than, (*b*) equal to, and (*c*) less than the weight of the contained liquid.

106. A bicycle pump is full of air at 15 pounds per square inch, length of stroke is 12 inches; at what part of stroke does air begin to enter tire at 40 pounds per square inch above atmospheric pressure? Assume the compression to take place without rise of temperature.

107. The density of mercury at 0° C. is 13.5956 grams per cubic centimeter. Calculate the value in dynes per square centimeter of standard atmospheric pressure, namely 76 cm. of mercury at 0° C., the value of gravity being 980.61 cm. per second per second.*

108. Calculate the height of the *homogeneous atmosphere;* that is, assuming that the atmosphere has a uniform density of 0.00129 grams per cubic centimeter throughout, calculate the depth which would produce standard atmospheric pressure.

109. The air in a 1,000 c.c. bulb is reduced to a volume of 0.02 c.c. when the pressure becomes 240 mm. of mercury. What is the original pressure of the air in the bulb?

110. The clearance space behind the piston of an air compressor when the piston is at the end of its stroke is $\frac{1}{50}$ of the *volume swept by piston* during the stroke. What is the greatest pressure that can be produced in a compressed air reservoir by this compressor, the compression of the air in the cylinder being assumed to be without change of temperature.

111. The piston of an air pump is 0.01 inch from the bottom of the cylinder when it is at the end of its stroke, and the pressure of the air in the clearance space is then at atmospheric pressure. The length of stroke is 6 inches. What is the lowest vacuum which can be produced by the pump?

112. A glass bulb weighs 75.405 g. when filled with air at standard temperature and pressure. It weighs 74.309 grams when the air is pumped out. It weighs 74.422 g. when filled with hydrogen at the same temperature and pressure. What is the specific gravity of hydrogen referred to air?

113. What is the net lifting capacity of a balloon containing 400 cu. m. of hydrogen and its material weighing 250 kg.? (Weight of a cu. m. of air is 1,200 g.; weight of a cu. m. of hydrogen is 90 g.)

114. How much work in foot-pounds is required to pump 10,000 cubic inches of water into a reservoir in which the pressure stands at the constant value of 150 pounds per square inch?

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115. The velocity of a jet of water is 200 feet per second, what is the kinetic energy of the water (in foot-pounds per cubic inch)? One cubic inch of water has a mass of .0376 pound.

116. Calculate velocity of efflux of kerosine from a vessel in which the pressure is 52 pounds per square inch, density of kerosine being 0.03 pound per cubic inch.

117. A street water main 7 inches inside diameter has a throat 3 inches inside diameter. The flow of water through the pipe is 2,500 cubic inches per second, and the pressure in the 7 inch pipe is 90 pounds per square inch. What is the pressure in the throat, ignoring friction?

118. If 50 c.c. per second of mercury flows through a tube 1 cm. in diameter, in which tube there is a throat 0.66 in diameter, what is the difference in pressure in throat and full-sized pipe?

Remark. — Mercury approximates more nearly to the ideal frictionless incompressible fluid than any other known liquid.

119. Quadrupling the pressure head is found to make the discharge through a pipe three times as great. That is, with pressure head p the discharge of water through a given pipe is q units volume per second, with pressure head $4p$ the discharge is observed to be $3q$. What portion of the pressure is lost by viscous friction and what portion by eddy friction in each case, loss of head by eddy friction being assumed proportional to the square of the discharge?

Remark. — Calling loss of head unity in first case and representing by v the loss of head by viscous friction, and by e the loss of head by eddy friction we have

$$1 = v + e \qquad \text{(i)}$$

When discharge becomes three times as great the total loss of head is 4 of which the loss of head due to viscous friction is $3v$ and the loss of head due to eddy friction is 3^2e so that

$$4 = 3v + 9e \qquad \text{(ii)}$$

From equations (i) and (ii) the values of v, e, and of $3v$ and $9e$, which are the required answers to the problem, may be found.

120. The coefficient of viscosity of water at ordinary room temperature is 0.0178 c. g. s. units (see Art. 187 Proposition). Two very smooth surfaces 100 cm. × 100 cm. have a layer of water between them of thickness 0.01 cm. How much force would be required to make one surface glide over the other at a velocity of 100 cm./sec?

CHAPTER IX.

TEMPERATURE; THERMAL EXPANSION.

121. Reduce to Fahrenheit readings the following centigrade temperatures:

45°, 12° and – 20°.

122. Reduce to centigrade readings the following Fahrenheit temperatures:

212°, 72°, 32° and – 30°.

123. At what temperature will both Fahrenheit and centigrade thermometers give the same reading?

124. An amount of gas at 15° C. occupies 120 c.c.; find its volume at 87° C., the pressure remaining constant.

125. A volume of hydrogen at 11° C. measures 4 liters. The gas is heated until the volume is found to have increased to 5 liters. Find the temperature of the hydrogen at this volume, the pressure remaining constant.

126. At standard steam temperature the pressure of the air in an air thermometer bulb is 766 mm. When the bulb is placed in a bath of melting lead the pressure of the air in the bulb is 1,440 mm. What is the temperature centigrade of melting lead, reckoned from ice-point?

127. What is the precise air thermometer temperature reckoned from ice point, corresponding to 45° C. mercury-in-glass? (See table, Art. 228.)

128. An iron steam pipe 1,000 feet long at 0° C. ranges in temperature from – 20° C. to 115° C. What must be the range of motion of an expansion joint to provide for expansion?

129. A brass rod is found to measure 100.019 cm. at 10° C. and 100.19 cm. at 100° C. Find the mean coefficient of linear expansion of brass between 10° and 100°.

130. A steel shaft is 20 inches in diameter at 70° F. A steel collar is to be shrunk upon this shaft. The collar is to be heated to 650° F. and have at that temperature an inside diameter of 20.01 inches. Required the inside diameter to which the collar must be turned in the shop, shop temperature being 70° F. Coefficient of linear expansion of steel is 0.0000066 per Fahrenheit degree.

131. A glass flask holds 842 grams of mercury at 0° C. How many grams of mercury will overflow if the whole is heated to 100° C.?

CHAPTER X.

CALORIMETRY; CHANGES OF STATE.

132. How many calories are there in one B.T.U.; how many ergs? *

133. Niagara Falls is 165 feet high. Calculate the rise in temperature of water due to the energy of fall.

134. A silver calorimeter weighing 50 gm. contains 500 gm. of water at 16°; a piece of silver weighing 65 gm. is heated to 100° and then plunged into the water; the resulting temperature is 16°.50. Find the specific heat of silver.

135. Fifteen pounds of lead at 0° C. is placed in a vessel containing 4 pounds of water at 100° C. To what temperature does the mixture settle?

See article 241 for table of specific heats.

136. Into 12 kg. of water at 30° C. are dropped, at the same instant, 1 kg. of copper at 100° C., 1.2 kg. of zinc at 60° C., and 1.5 kg. of ice at $-20°$ C. Find the resultant temperature. The specific heat of ice is 0.51.

137. Find the amount of heat necessary to raise 3 kg. of lead at 10° C. to the melting point, and then to melt it.

For tables of melting points and of heats of fusion see Arts. 243 and 248.

138. A mass of iron weighing 160 g. is heated to 100° C. and dropped into an ice calorimeter. The mass of ice melted is 22.4 g. Find the specific heat of iron.

139. How much ice per day (24 hours), is required to reduce the temperature of an air blast from 25° C. to 17° C., the blast furnishing 1 cubic meter per second; the air being measured at 760 mm. and at a temperature of 25° C.?

The density of air at 760 mm. and 25° C. is 0.00118 gram per cubic cm., and the specific heat of air (at constant pressure) is 0.2375 calorie per gram.

140. A closed bottle is full of dry air at 760 mm. pressure and 50° C. A small quantity of water is introduced into the bottle. Required the total pressure of air and water vapor.

For table of maximum pressures of water vapor see Art. 244.

141. A gas is measured over pure water at a temperature of 15° C. and the observed pressure is 752 mm. What would the pressure of the given amount of gas be if it occupied the same volume dry, that is free from admixture of water vapor?

CHAPTER XI.

PROPERTIES OF GASES.

142. Find the volume of 2 pounds of oxygen at a pressure of 3 atmospheres and a temperature of 27° C., the volume of 1 pound of oxygen at 0° C. and 1 atmosphere being 11.204 cubic feet.

143. One volume of oxygen, two atoms in the molecule, combines with two volumes of hydrogen, two atoms in the molecule, to form H_2O. How many volumes of H_2O are formed?

144. One volume of chlorine, two atoms in the molecule, combines with one volume of hydrogen, two atoms in the molecule, to form HCl. How many volumes of HCl are formed?

145. Calculate the square root of mean square of the velocities of hydrogen molecules at 0° C.; also of oxygen molecules at 0° C.

Remark. — The product nm [see equation (138)] is the density of a gas, therefore easily observed for a given pressure p, so that ω may be calculated. When p is 760 mm. (1,013,373) dynes/cm.^{2}) the density of hydrogen at 0°C. is 0.00008954 grams per cubic centimeter, and the density of oxygen is sixteen times as great.

146. Air at 15 pounds per square inch pressure and at 20° C. is pumped into a bicycle tire at a pressure of 40 pounds per square inch (above atmospheric pressure). Find the temperature of the compressed air, assuming the compression to be isentropic, and find at what part of the stroke the air reaches 40 pounds per square inch (above atmospheric pressure). Compare problem 106.

147. Solve problem 110 on the assumption that the compression of the air in the compressor cylinder is isentropic.

148. Air at 200 pounds per square inch (above atmospheric pressure) used to drive an air engine, exhausts at atmospheric pressure, 15 pounds per square inch. Required the temperature of the high pressure air in order that there may be no possibility of frost forming in the exhaust ports of the engine, the expansion of the air in the engine being assumed to be isentropic.

149. The atmospheric pressure at the ground is 755 mm. At a distance of 2,000 ft. above the ground the pressure is 695 mm. The temperature of the air at the ground is 31° C. Find the value of the dew point of the air at the ground in order that a rising column of air would form a cloud at 2,000 ft. above the ground.

Remark. — In this problem neglect the influence of the water vapor upon the law of adiabatic expansion of the air. Note that the pressure of the water vapor is decreased in the ratio 755 to 695 so that the air will have to be cooled below the temperature which expresses the dew point at the ground.

The following table showing relation between dew point and vapor pressure is needed in the solution of this problem.

Dew Point.	Vapor Pressure.
23° C.	20.9 mm.
24° C.	22.2 mm.
25° C.	23.5 mm.
26° C.	25.0 mm.

CHAPTER XII.

THERMODYNAMICS.

150. Pressure in Watt's diagram is represented by ordinates 10 pounds per square inch per inch of ordinate and volumes are represented by abscissas 100 cubic inches per inch of abscissa. What amount of work (in foot-pounds) is represented by each square inch of area of a process curve?

151. A cylinder 12 inches long (in the clear) and 10 inches in diameter contains air at 15 pounds per square inch. The air is compressed isothermally. Plot on cross-section paper the curve showing the pressure for each position of the moving piston while the piston moves from a point 12 inches from the end of the cylinder to a point 6 inches from the end of the cylinder, and calculate the work in foot-pounds done during the compression.

Remark. — Calculate this work by determining the area under the plotted curve. To determine this area divide it up into six or eight vertical strips and calculate area of each approximately.

152. The gas in the above cylinder is compressed isentropically. Plot the curve, showing the pressure for each position of the piston in moving from a point 12 inches from the end of the cylinder to a point 6 inches from the end of the cylinder, and calculate the work in foot-pounds done during the compression.

153. A perfect engine taking steam at a temperature of 160° C. and exhausting it at 70° C. would have what efficiency? A good steam engine working between these temperatures has an efficiency equal to 75% of the efficiency of a perfect engine. What is the actual efficiency of the steam engine?

154. The mixture of air and gas in a gas engine reaches a temperature of say 1100° C. when it is ignited, and the temperature of this mixture is reduced to say 600° C. by expansion in the engine. What would be the efficiency of a perfect engine working between these temperatures?

155. A gas engine takes 100 cubic inches of hydrogen and 1,000 cubic inches of air (20% oxygen and 80% nitrogen) at 25° C. and 15 lbs. per square inch pressure. This mixture is then compressed isentropically to 110 cubic inches. What is the pressure of the mixed gas before explosion. (Assume k equal to 1.4 for this mixture and also for the following mixture of nitrogen, oxygen, and water vapor.)

When the mixture explodes the temperature rises to say 1200° C. What is the pressure after the explosion. (Do not neglect Avogadro's principle.)

The exploded mixture is then expanded again to 1,100 cubic inches. Find temperature and pressure at exhaust, assuming the expansion to be isentropic.

Find the efficiency of the engine, assuming it to be a reversible engine.

156. A refrigerating machine is required to take heat from a room at − 10° C. and deliver it to a water cooler at 35° C. Assuming the refrigerating machine to be a perfect engine, calculate the work required to freeze 1 kilogram of ice from ice water. Assuming the refrigerating machine to take $1\frac{1}{2}$ times as much work as a perfect engine, calculate the work required to freeze 1 kilogram of ice from ice water. Express the result in horse-power-hours and in kilowatt-hours.

157. A perfect engine takes compressed air at 25° C. and 150 pounds per square inch (above atmospheric pressure). At what cut-off must the engine be adjusted in order that the temperature of the exhaust may be − 20° C.?

158. Calculate the entropy increment (thermodynamic degeneration) represented by the direct conversion of 500 joules of work into heat at 400° C., into heat at 200° C., into heat at 0° C. Express the result in calories per degree. One calorie = 4.2 joules.

159. Calculate the entropy increment (thermodynamic degeneration) represented by the direct transfer of 500 joules of heat from 400° C. to 200° C., from 200° C. to 0° C.

160. Calculate the entropy of ten grams of water in the form of saturated steam at 100° C. Base the calculation upon assumed constancy or specific heat of water from 0° to 100° taking ice water as the zero state. Express the result in calories per degree.

Remark. — Divide the rise of temperature from 0° C. to 100° C. into, say, 20 equal steps and calculate the quotient $\Delta H/T$ for each step.

CHAPTER XIII.

TRANSFER OF HEAT.

161. A boiler shell is made of iron 1 cm. thick and 20,000 calories per hour flow through each square centimeter of the shell. What is the temperature difference between inner and outer surface of shell? What would this temperature difference be with a copper shell of the same thickness. Use the following values of thermal conductivities: Copper, 0.80; iron, 0.12; in calories/cm. sec. degree C.

Remark. — Thermal conductivities of metals fall off rapidly with rise of temperature.

162. The inside surface of the window glass in a house is at a temperature of 15° C., the outside surface is at a temperature of – 10° C., and the glass is 4 mm. thick. Calculate the heat in calories which, during 24 hours, flows out of the house by conduction through a total window glass area of 20 square meters. Reduce the result to kilograms of coal at 6,000 calories per gram.

Take 0.0015 for thermal conductivity of glass.

163. A stream of cold air 1 inch × 1 inch section blows into a room from outdoors at a velocity of 10 feet per second. The external temperature is –10° C. and this air has to be heated to 20° C. in the room. Calculate the heat in calories required to heat this stream of air from –10° C. to 20° C. for 24 hours and find the area of window glass which, under conditions of previous problem, would cause the loss of this amount of heat.

Assume that the air in above stream weighs 0.0075 gram per cubic inch. The specific heat of air at constant pressure is 0.24 calorie per gram.

164. A metal vessel containing 2,520 grams of water has a flat face 30 × 30 cm. which is pressed against the outside of a furnace wall made of brick, and the temperature of the water is observed to rise 1.5° C. in twenty minutes. The wall is 30 cm. thick, the inner face of the wall is at a temperature of 1500° C. and the outer face of the wall is at 25° C. What is the thermal conductivity of the material of which the wall is made?

165. An electric wire is heated to a temperature 20° above surroundings by a given electric current. How great a rise in temperature will be produced if the current is doubled so that heat is generated in the wire four times as fast as before?

166. When the steam pressure in a heating plant is increased the flow of steam through the system is more rapid and all the radiators are kept more nearly at boiler temperature. Suppose, however, that the radiators are always at boiler temperature, then how many times as much heat will be delivered to a building by increasing the steam pressure from 5 pounds per square inch to 100 pounds per square inch?

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